Ternary Addition. We define a binary operation '$*$' on the real numbers so that for every a, b, c, $(a * b) * c = a + b + c$. Prove that $*$ is $+$.

Solution. First we prove that $0 * 0 = 0$. We have:

$$
\begin{aligned}
2\,(0 * 0) = (0 * 0) + (0 * 0) + 0 = ((0 * 0) * (0 * 0)) * 0 = \\
(0 + 0 + (0 * 0)) * 0 = (0 * 0) * 0 = 0 + 0 + 0 = 0\,,
\end{aligned}
$$

i.e.: $2\,(0 * 0) = 0$, hence $0 * 0 = 0$, QED.

Then we prove that $a * b = a + b$:

$$a * b = (0 + 0 + a) * b = ((0 * 0) * a) * b = (0 * 0) + a + b = 0 + a + b = a + b\,.$$

This completes the proof.

$\square$

Additional question. From the proof we see that we can replace in the problem "real numbers" with other kinds of numbers (rational, integral, etc.) and the result is still true. But are there any exceptions? Is there any kind of numbers for which $(a * b) * c = a + b + c$ for every a, b, c does not imply that $*$ is $+$?

Answer. Yes, for instance, integers modulo 2, and $a * b = 1 + a + b \mod 2$.

Miguel A. Lerma - 2/27/2007