Split Squares. Prove that there are infinitely many squares not multiple of 10 whose representation in base 10 can be split into two squares. For instance $7^2 = 49$ can be split $4|9$, where 4 and 9 are squares ($4 = 2^2$, $9 = 3^2$); $13^2 = 169$ can be split $16|9$, again two squares, etc. (we exclude multiples of 10 in order to avoid trivial answers like the infinite sequence $49 = 4|9$, $4900 = 4|900$, $490000 = 4|90000$, etc.).

Solution. The fact that the decimal representation of a square z^2 (not a multiple of 10) is the concatenation of two squares x^2 and y^2 can be expressed with the following system of equation and inequality:

$$\begin{aligned} &10^n x^2 + y^2 = z^2 \\ &10^{n-1} < y^2 < 10^n \,, \end{aligned} \tag{1}$$

where x, y, z, n must be positive integers and y and z are not multiple of 10. So we need to prove that (1) has infinitely many solutions. In fact we will prove more, namely that for any given positive integer x, (1) has infinitely many solutions. So in the following we assume that x is any fix given positive integer.

We start by rewriting the equation in the following way:

$$10^n x^2 = z^2 - y^2 = (z + y)(z - y).$$

Since the left hand side is even, y and z must have the same parity, so the two factors on the right must be even and we can write $z + y = 2p$, $z - y = 2q$ for some positive integers p and q. Then we have $z = p + q$, $y = p - q$, and $10^n x^2 = 4pq$, so $q = 10^n x^2/(4p)$. Hence the inequality can be written like this:

$$10^{(n-1)/2} < p - \frac{10^n x^2}{4p} < 10^{n/2} \,.$$

The expression $f(p) = p - 10^n x^2/(4p)$ is an increasing function of p, and verifies $f(10^{n/2} b_1/2) = 10^{(n-1)/2}$ and $f(10^{n/2} b_2/2) = 10^{n/2}$, where

$$b_1 = 1/\sqrt{10} + \sqrt{1/10 + x^2} \quad \text{and} \quad b_2 = 1 + \sqrt{1 + x^2} \,.$$

So the inequality becomes

$$\frac{10^{n/2}}{2} b_1 < p < \frac{10^{n/2}}{2} b_2 \,.$$

Taking decimal logarithms we get

$$\frac{n}{2} + \log_{10} b_1 - \log_{10} 2 < \log_{10} p < \frac{n}{2} + \log_{10} b_2 - \log_{10} 2$$

or equivalently

$$n < 2 \log_{10} p + \alpha < n + \beta \,,$$

where, $\alpha = 2 \log_{10}(2/b_1)$, $\beta = 2 \log_{10}(b_2/b_1)$. We note that α and β depend only on x, but not on p or n, and also that $\beta > 0$. Also recall that $4p$ must be a divisor of $10^n x^2$, and $p \pm q$ should not be a multiple of 10. These conditions are met if we set $n > 2$ and $p = 5^k$ for some $0 \le k < n$. Then the inequality becomes

$$n < 2k \log_{10} 5 + \alpha < n + \beta \,,$$

or equivalently

$$n = \lfloor 2k \log_{10} 5 + \alpha \rfloor \,,$$
$$0 < \{\!\{ 2k \log_{10} 5 + \alpha \}\!\} < \beta \,,$$

where $\lfloor t \rfloor$ = integer part of t, $\{\!\{t\}\!\} = t - \lfloor t \rfloor$ = fractional part of t. Since $2 \log_{10} 5 > 1$, the condition $k < n$ will be satisfied for every k large enough. On the other hand since the integer multiples of an irrational number are dense modulo 1, and $2 \log_{10} 5$ is indeed irrational, we have that the fractional part of $2k \log_{10} 5$ is in $(0, \beta)$ for infinitely many values of k. So since all the conditions are satisfied for infinitely many values of k, we have that (1) has infinitely many solutions.

The argument used here can be used to search numerically for specific solutions of (1). The idea is to pick any positive integer x and assign values $1, 2, 3, \dots$ to k checking whether the following conditions are verified:

$$n = \lfloor 2k \log_{10} 5 + \alpha \rfloor > k \,,$$
$$0 < \{\!\{ 2k \log_{10} 5 + \alpha \}\!\} < \beta \,,$$

Example: First we pick any positive value for x, say $x = 1$. Next we compute $2 \log_{10}(5) = 1.397940008\dots$, $\alpha = 0.3317713906\dots$, $\beta = 0.4952627696\dots$. Finally we search for values of k such that

$$n = \lfloor 1.397940008k + 0.3317713906 \rfloor > k \,,$$
$$0 < \{\!\{ 1.397940008k + 0.3317713906 \}\!\} < 0.4952627696 \,.$$

For instance, for $k = 2$ we have $1.397940008k + 0.3317713906 = 3.127651407$, so $k = 2$ satisfies the conditions, yielding the solution $n = 3$, $p = 5^2 = 25$, $q = 10^3/(4 \cdot 25) = 10$, $y = 25 - 10 = 15$, $z = 25 + 10 = 35$. So $y^2 = 225$, $z^2 = 1225$. Hence $35^2 = 1225 = 1|225$ can be split into $1 = 1^2$ and $225 = 15^2$.

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