Series involving e

Problem. Find the sum of the following series:

$$\sum_{n=1}^{\infty} \left\{ e - \left(1 + \frac{1}{n}\right)^n \right\}$$

Solution. The series diverges to $+\infty$.

Proof. Let $A_n = n \ln\bigl(1 + \frac{1}{n}\bigr)$. Using the following inequality for $t > 0$:

$$\ln(1+t) \le t - \frac{t^2}{2} + \frac{t^3}{3},$$

and setting $t = \frac{1}{n}$, we obtain

$$a_n \le 1 - \frac{1}{2n} + \frac{1}{3n^2}.$$

Hence

$$\left(1 + \frac{1}{n}\right)^n = e^{a_n} \le e^{\,1 - \frac{1}{2n} + \frac{1}{3n^2}} = e\, e^{-\frac{1}{2n} + \frac{1}{3n^2}}.$$

therefore

$$e - \left(1 + \frac{1}{n}\right)^n \ge e\bigl(1 - e^{-\frac{1}{2n} + \frac{1}{3n^2}}\bigr).$$

For small $z > 0$, $1 - e^{-z} \ge z - \frac{z^2}{2}$. Taking $z = \frac{1}{2n} - \frac{1}{3n^2}$, we get

$$1 - e^{-\frac{1}{2n} + \frac{1}{3n^2}} \ge \left(\frac{1}{2n} - \frac{1}{3n^2}\right) - \frac{1}{2}\left(\frac{1}{2n}\right)^2 = \frac{1}{2n} - \frac{11}{24}\frac{1}{n^2}.$$

Thus, for all sufficiently large n,

$$e - \left(1 + \frac{1}{n}\right)^n \ge \frac{c}{n} \quad \text{for some constant } c > 0 \text{ (for instance, } c = \tfrac{e}{3}\text{)}.$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test we conclude that

$$\sum_{n=1}^{\infty} \left\{ e - \left(1 + \frac{1}{n}\right)^n \right\} = +\infty.$$

Asymptotic check. Expanding $\left(1 + \frac{1}{n}\right)^n = e^{1 - \frac{1}{2n} + o(1/n^2)} = e\left(1 - \frac{1}{2n} + o(1/n^2)\right)$, we find

$$e - \left(1 + \frac{1}{n}\right)^n \sim \frac{e}{2n},$$

confirming that the series diverges like a harmonic series. □

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