Probability Problem With Roulette Wheels

Problem. We have N roulette wheels, each with the same probability p of stopping at zero.

1. We spin all N roulette wheels simultaneously. What is the expected number of spins needed until all of them stop at zero simultaneously?
2. We spin the first roulette wheel until it stops at zero. Then we do the same with the second one, and so on, until all roulette wheels have stopped at zero. What is the expected total number of spins required until all of them have stopped at zero?
3. We randomly pick a roulette wheel and spin it. We continue selecting roulette wheels at random and spinning them. What is the expected number of spins needed until n $(1 \leq n \leq N)$ distinct wheels have each stopped at zero at least once?
4. We spin all N roulette wheels simultaneously. After each spin, we continue spinning only the wheels that did not stop at zero. What is the expected number of spins needed until at least n $(1 \leq n \leq N)$ of them have stopped at zero?

Solution.

1. **All wheels must land on zero simultaneously.** If an event has probability p, the expected number of repetitions until the event happens is $\frac{1}{p}$. In our case, the probability that all the wheels stop at zero simultaneously is p^N, hence the expected number of spins T until success is
$$\boxed{\mathbb{E}[T] = \frac{1}{p^N}}.$$

2. **Sequentially spin each wheel until it lands on zero.** By the linearity of the expected value, the answer is just the sum of the expected number of times we must spin each wheel until it stops at zero. For each wheel the expected time is $\frac{1}{p}$, so for the N wheels it will be
$$\boxed{\mathbb{E}[T] = \frac{N}{p}}.$$

3. **Randomly choose a wheel to spin until n of them have stopped at zero at least once.** This is a version of the *coupon collector's problem*.

   Let T_n be the number of times taken until n wheels are stopped at zero at least once, and let $\mathbb{E}[T_n]$ be its expected value, i.e., the expected number of times we must spin wheels until n of them have stopped at zero at least once. Then $\mathbb{E}[T_1] = \frac{1}{p}$, and $\mathbb{E}[\mathbb{T}_{k+1}] - \mathbb{E}[T_k]$ = expected number of spins to get one more wheel stopping at zero for the first time after k of them already did it. The probability of picking a wheel that has not yet stopped at zero is $\frac{N-k}{N}$, and the probability of it stopping at zero is the product $\frac{N-k}{N}\,p$, so the expected time for that event to occur is $\frac{N}{(N-k)\,p}$. Hence $\mathbb{E}[T_n] = \frac{N}{p}\sum_{k=0}^{n-1}\frac{1}{N-k}$, and we get

$$\boxed{\mathbb{E}[T_n] = \frac{N}{p}(H_N - H_{N-n})},$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ = nth harmonic number.

Added: Asymptotic approximations for case 3. Next, we show a few approximations of $\mathbb{E}[T_n]$ than can be obtained using the asymptotic expansion $H_n = \ln(n) + \gamma + \frac{1}{2n} + O(\frac{1}{n^2})$ as $n \to \infty$:

(a) If $n = N$ then $\mathbb{E}[T_N] = \frac{1}{p}\left\{N\ln(N) + \gamma + \frac{1}{2} + O(\frac{1}{N})\right\}$.

(b) More generally $\mathbb{E}[T_n] = \frac{1}{p}\left\{-N\ln(1-\frac{n}{N}) - \frac{n}{2(N-n)} + O(\frac{1}{N})\right\}$, useful when both N and $N-n$ are large. The approximation $\mathbb{E}[T_n] \approx -\frac{N}{p}\ln(1-\frac{n}{N})$ resembles a process of radioactive decay ($n = N(1-e^{-\lambda t})$, with $t = \mathbb{E}[T_n]$, $\lambda = \frac{p}{N}$).

(c) The expression can be rewritten:

$$\mathbb{E}[T_n] = \frac{1}{p}\sum_{i=0}^{\infty}\left(\frac{1}{N^i}\sum_{k=1}^{n-1}k^i\right) = \frac{n}{p}\left\{1 + \frac{n-1}{2N} + \frac{(2n-1)(n-1)}{6N^2} + \cdots\right\},$$

which can be used to approximate $\mathbb{E}[T_n]$ for large N and small n/N.

4. **Spin all wheels simultaneously, but stop spinning the ones that have already stopped at zero.** This case is equivalent to a sequential independent geometric waiting times, where in each round the number of active wheels decreases.

   Let X_k be the number of rounds between the k-th and $k+1$-th wheel stopping at zero (with X_0 being the rounds until the first success). In each round we spin $N-k$ active wheels. The probability that at least one of these active wheels stops at zero in a single round is:

   $$P_k = 1 - (1-p)^{N-k}.$$

   Therefore, the expected number of rounds between the k-th and $k+1$-th wheels stopping at zero is:

   $$\mathbb{E}[X_k] = \frac{1}{1-(1-p)^{N-k}}.$$

   Summing over $k = 0$ to $n-1$, the expected total number of rounds until at least n wheels have stopped at zero is:

   $$\boxed{\mathbb{E}[T_n] = \sum_{k=0}^{n-1}\frac{1}{1-(1-p)^{N-k}}}.$$

Added: Asymptotic Approximations for case 4.

(a) **Small p approximation:** When $p \ll 1$, we can approximate

$$(1-p)^m \approx e^{-pm} \quad \text{for } m = N-k.$$

Therefore,

$$\mathbb{E}[T_n] \approx \sum_{k=0}^{n-1} \frac{1}{1-e^{-p(N-k)}}.$$

(b) **Large N, moderate n:** If N is small and n is moderate, we can approximate the sum by an integral. Define $x = k/N$, then:

$$\mathbb{E}[T_n] \approx N \int_0^{n/N} \frac{1}{1-(1-p)^{N(1-x)}}\, dx.$$

This integral can be evaluated numerically, or, if pN is also small, further approximated using Laplace's method, leading to (see appendix):

$$\mathbb{E}[T_n] \approx \frac{1}{p} \log\left(\frac{N}{N-n}\right) \qquad \text{for large } N \text{ and } p \ll 1/N.$$

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Appendix: Laplace-Type Asymptotic Approximation for $\mathbb{E}[T]$ in case 4.

Recall the integral approximation derived above:

$$\mathbb{E}[T_n] \approx N \int_0^{n/N} \frac{1}{1-(1-p)^{N(1-x)}}\,dx.$$

For small p and large N, we use the approximation:

$$(1-p)^{N(1-x)} \approx e^{-pN(1-x)} \quad \Rightarrow \quad \frac{1}{1-(1-p)^{N(1-x)}} \approx \frac{1}{1-e^{-pN(1-x)}}.$$

Next, change variables:

$$u = N(1-x) \quad \Rightarrow \quad x = 1 - \frac{u}{N}, \quad dx = -\frac{1}{N}\,du.$$

As x goes from 0 to n/N, u goes from N to $N-n$. Rewriting the integral:

$$\mathbb{E}[T] \approx \int_{N-n}^{N} \frac{1}{1-e^{-pu}}\,du.$$

If pN is small, then pu is small and can we can use the Taylor approximation:

$$e^{-pu} = 1 - pu + \frac{(pu)^2}{2} + \cdots \quad \Rightarrow \quad 1-e^{-pu} \approx pu \quad \Rightarrow \quad \frac{1}{1-e^{-pu}} \approx \frac{1}{pu}.$$

Thus:

$$\mathbb{E}[T_n] \approx \int_{N-n}^{N} \frac{1}{pu}\,du = \frac{1}{p}\int_{N-n}^{N} \frac{1}{u}\,du = \frac{1}{p}\left(\log N - \log(N-n)\right) = \frac{1}{p}\log\left(\frac{N}{N-n}\right).$$

Hence:

$$\boxed{\mathbb{E}[T_n] \approx \frac{1}{p}\log\left(\frac{N}{N-n}\right)} \qquad \text{for large } N \text{ and } p \ll 1/N.$$