John's PIN. John's PIN for his teller machine consists of a 6-digit number '`abcdef`'. He has bad memory, but does not want to write it down just in case someone finds it. So he breaks the number in two 3-digit numbers '`abc`' and '`def`', and with his pocket calculator finds the quotient `abc/def`, which he writes down. A few days later he needs the PIN, and as expected he cannot remember it, but he remembers that he wrote down the quotient between the two halves of the number in a piece of paper: '`abc/def`= 0.195323246'. So he takes his pocket calculator (with only basic arithmetic operations and the inversion '$1/x$' key), punches the keys for a few seconds, scratches some numbers on a piece of paper for a few seconds more and gets the original 6-digit number. What is that number and how did he find it so quickly?

Solution. First note that if p/q and r/s are two fractions with 3-digit numerator and denominator whose values coincide up to the 9th decimal place then we have $|p/q - r/s| \leq 10^{-9}$ $\Rightarrow |ps - qr| \leq 10^{-3} \Rightarrow ps = qr$ (because they are integers), hence $p/q = r/s$.

A way to solve the problem is to try all possible fractions of the form `abc/def` until getting one whose decimal representation coincides with the given one, but that would take too long and it would be very unlikely to find it in a few seconds even with a calculator. A more efficient way is to represent the given decimal number r as a continued fraction of the form

$$r = [a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

where the a_k's (*partial quotients*) are integers. They can be computed using the recursive relations $r_0 = r$, $a_n = \lfloor r_n \rfloor$, $r_{n+1} = 1/(r_n - a_n)$ $(n \geq 0)$. Rational numbers have a finite continued fraction, so eventually $a_n = r_n$ and a that point we stop the computation. Due to rounding by the calculator in practice $r_n - a_n$ may not become exactly zero, but we can recognize when to stop when $r_n - a_n$ becomes very small. For $x = 0.195323246$ we get $a_0 = 0$, $a_1 = 5$, $a_2 = 8$, $a_3 = 2$, $a_4 = 1$, $a_5 = 5$. At this point we have $r_5 - a_5 = 0.00011475\dots$, which is very small, so we write

$$x = [0; 5, 8, 2, 1, 5] = \cfrac{1}{5 + \cfrac{1}{8 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{5}}}}} = \frac{142}{727}.$$

Finally we check the result:

$$\frac{142}{727} = 0.195323246\dots$$

This is the only fraction with 3-digit numerator and denominator whose decimal representation coincides with the given one, so John's PIN is 142727.

Miguel A. Lerma - 10/16/2003