Covering the Real Line with Disjoint Closed Intervals

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Question. Is it possible to cover the real line $\mathbb{R}$ with non-degenerate, disjoint, closed, bounded intervals?

Answer. No, it is not possible to cover $\mathbb{R}$ with non-degenerate, disjoint, closed, bounded intervals.

Note that a non-degenerate closed interval have a non-empty open interior, hence it contains some rational number, and disjoint intervals cannot contain the same rational number. It follows that any collection of disjoint, non-degenerate intervals is countable, hence it suffices to prove the following slightly more general statement:

Statement. It is not possible to cover $\mathbb{R}$ with countably many (not necessarily non-degenerate), disjoint, closed, bounded intervals.[1]

Proof. Assume $\{I_i\}$ is a countable set of disjoint closed, bounded intervals, $I_i = [a_i, b_i]$ with $a_i \leq b_i$. Let U be their union. We will show that U can never be the whole real line $\mathbb{R}$.

It is clear that a finite set of closed, bounded intervals cannot cover $\mathbb{R}$, so we can consider only the case in which the set of intervals is countable infinite. Next, set

$$J_i := (a_i, b_i) \text{ for every i}, \quad V := \bigcup_{i=0}^{\infty} J_i, \quad T := \mathbb{R} \setminus V.$$

The set V is open, so T is closed. Moreover, the set $E := \{a_i, b_i : i \in \mathbb{N}\} \subseteq T$ is countable.

We claim that T is a *perfect* set, i.e., a closed set with no isolated points. Let's prove this.

We already have that T is closed, so it remains to show that it does not have isolated points. In fact, assume $p \in T$ is an isolated point in T, so for every $\varepsilon > 0$ small enough we have $(p-\varepsilon, p+\varepsilon) \cap T = \{p\}$, meaning the interval $(p-\varepsilon, p+\varepsilon)$ lies entirely within $\mathbb{R} \setminus T = \bigcup J_i$, except for the point p. Thus, some $J_i = (a_i, b_i)$ must meet $(p - \varepsilon, p + \varepsilon)$. Since $p \notin J_i$, we cannot have $a_i < p < b_i$, hence, either $p - \varepsilon < b_i \leq p$, or $p \leq a_i < p + \varepsilon$. In either case, $(p - \varepsilon, p + \varepsilon)$ must contain an endpoint $x_i = a_i$ or b_i of J_i, and $|p - x_i| < \varepsilon$.

This implies that p is a limit point of the endpoint set E, and since $E \subseteq T$, p is also a limit point of T, contradicting the assumption that p is isolated in T.

Therefore, T has no isolated points and is perfect.

[1] See [1] for a related discussion about covering a non-closed interval with closed intervals.

In Euclidean space, it is a well-known result that every perfect set is uncountable. On the other hand, E is countable, hence $E \subsetneq T$.

Finally, note that U is the disjoint union of V and E, hence:

$$\mathbb{R} \setminus U = (\mathbb{R} \setminus V) \setminus E = T \setminus E \neq \emptyset.$$

Thus, U does not cover $\mathbb{R}$, and this completes the proof. $\square$

Additional considerations

How much of $\mathbb{R}$ can be covered with disjoint, non-degenerate, bounded, closed intervals? While we cannot cover the whole real line with disjoint, non-degenerate, closed, bounded intervals, we *can* cover a *dense subset* of $\mathbb{R}$. Indeed, consider the poset of unions of such disjoint intervals, ordered by inclusion. Every chain in this poset has an upper bound given by the union of its elements. By Zorn's Lemma, there exists a maximal element. If its union were not dense in $\mathbb{R}$, we could add another interval in an open uncovered portion, contradicting maximality. Thus, this maximal union is dense in $\mathbb{R}$.

How about higher dimensions? The results obtained can easily be extended to $\mathbb{R}^n$ with the closed intervals replaced with convex, compact subsets of $\mathbb{R}^n$ with non-empty interior, such as non-degenerate, closed balls or cubes. Again, the "non-empty interior" condition can be replaced with "countably many," as we showed in the 1-dimensional case. Next, note that an infinite straight line in $\mathbb{R}^n$ will intersect each convex, compact subset of $\mathbb{R}$ in a bounded closed interval (or not intersect it at all), so a covering of $\mathbb{R}^n$ will induce a covering of that line by countably many disjoint, closed, bounded intervals, which we proved is impossible.

The Zorn's Lemma argument also applies to the poset of all families of convex, compact subsets of $\mathbb{R}^n$ with non-empty interior, ordered by inclusion. A maximal element in this poset will cover a dense subset of $\mathbb{R}^n$, but it will never be enough to cover the totality of $\mathbb{R}^n$.

References

[1] T. Tao, *Covering a non-closed interval by disjoint closed intervals*, blog post, October4, 2010. Available at `https://terrytao.wordpress.com/2010/10/04/covering-a-non-closed-interval-by-disjoint-closed-intervals/`.