A binary operation on rationals. Let $\circ$ be a binary operation defined on rational numbers with the following properties:

(1) Commutative: $a \circ b = b \circ a$.
(2) Associative: $a \circ (b \circ c) = (a \circ b) \circ c$.
(3) Idempotency of zero: $0 \circ 0 = 0$
(4) Distributivity of '+' respect to '$\circ$': $(a \circ b) + c = (a + c) \circ (b + c)$.

Prove that the operation is either $a \circ b = \max(a, b)$ or $a \circ b = \min(a, b)$.

Solution. First we note that every element is idempotent:

$$a \circ a = (0 + a) \circ (0 + a) = (0 \circ 0) + a = 0 + a = a\,.$$

We will say that "b absorbs a", written $a \preceq b$, if $a \circ b = b$. This is an order relation, since it is:

(1) Reflexive: $a \circ a = a \Rightarrow a \preceq a$.
(2) Antisymmetric: if $a \preceq b$ and $b \preceq a$ then $a \circ b = b$ and $a \circ b = a$, hence $a = b$.
(3) Transitive: if $a \preceq b$ and $b \preceq c$ then $a \circ b = b$ and $b \circ c = c$, hence $a \circ c = a \circ (b \circ c) = (a \circ b) \circ c = b \circ c = c$, so $a \preceq c$.

Some additional properties of $\preceq$ are:

(1) $a \preceq b \Rightarrow a + c \preceq b + c$. In fact, because of distributivity of $+$ respect to $\circ$ we have:

$$a \preceq b \;\Rightarrow\; a \circ b = b \;\Rightarrow\; (a + c) \circ (b + c) = b + c \;\Rightarrow\; a + c \preceq b + c\,.$$

(2) $a \circ b$ is the least upper bound of a and b respect to $\preceq$, i.e., $a \preceq a \circ b$ and $b \preceq a \circ b$, and if $a \preceq c$ and $b \preceq c$ then $a \circ b \preceq c$. In fact:

$$a \circ (a \circ b) = (a \circ a) \circ b = a \circ b \;\Rightarrow\; a \preceq a \circ b\,,$$

and similarly for $b \preceq a \circ b$. On the other hand, if $a \preceq c$ and $b \preceq c$ then $a \circ c = c$ and $b \circ c = c$, hence $a \circ b \circ c = a \circ c = c$, so $a \circ b \preceq c$.

Next, define $0^* = \{x \in \mathbb{Q} \mid 0 \preceq x\}$ (rational numbers absorbing 0). This set has the following properties:

(1) It contains zero: $0 \in 0^*$.
(2) It is closed respect to addition. i.e., $x, y \in 0^* \Rightarrow x + y \in 0^*$. In fact, if $x, y \in 0^*$ then $0 \preceq x$ and $0 \preceq y$, so $0 + x \preceq x + y$, i.e., $x \preceq x + y$, and by transitivity $0 \preceq x + y$.
(3) It is closed respect to multiplication by positive integers:

$$n \in \mathbb{Z}^+ \text{ and } x \in 0^* \;\Rightarrow\; nx = x \overset{(n \text{ times})}{+ \cdots +} x \in 0^*\,.$$

(4) It is closed respect to division by positive integers: If $x \in 0^*$ and $m \in \mathbb{Z}^+$, consider the $m+1$ points $x_k = kx/m$ for $k = 0, 1, \dots, m$. Let $u = x_0 \circ x_1 \circ \cdots \circ x_{m-1}$ and $v = x_1 \circ x_2 \circ \cdots \circ x_m = u + x/m$. We have

$$u \circ v = x_0 \circ x_1 \circ \cdots \circ x_m \,,$$

but $x_0 = 0$ and $x_m = x$, and by hypothesis $0 \preceq x$, i.e., $0 \circ x = x$, so

$$x_0 \circ x_1 \circ \cdots \circ x_m = x_1 \circ \cdots \circ x_m = v \,,$$

hence $u \circ v = v$, $u \preceq v$, $0 \preceq v - u = x/m$, i.e., $x/m \in 0^*$.

(5) As a consequence of the previous two results, 0^* is closed respect to multiplication by positive rational numbers.

(6) It contains some non-zero element: given two different rational numbers a, b, we have $a \preceq a \circ b$ and $b \preceq a \circ b$, hence $(a \circ b) - a$ and $(a \circ b) - b$ are in 0^*, and at least one is not zero.

(7) Either all its non-zero elements are positive, or they are all negative. This can be proved by contradiction: Assume 0^* contains both positive and negative elements, i.e., there are $x, y \in 0^*$, $x < 0$, $y > 0$. Since 0^* is closed respect to multiplication by positive rational numbers, we have that $x \cdot 1/(-x) = -1$ and $y \cdot 1/y = 1$ are in 0^*, so $0 \preceq -1$ and $0 \preceq 1$. Adding 1 to the first expression we get $1 \preceq 0$, and by antisymmetry $0 = 1$, which is absurd.

Since 0^* is closed respect to multiplication by positive rational numbers there are two possibilities, either 0^* consists of all positive rational numbers and zero, or it consists of all negative rational numbers and zero. Assume first that 0^* consists of all positive rational numbers and zero. Then $0 \preceq x \Leftrightarrow 0 \leq x$, or equivalently, $a \preceq b \Leftrightarrow a \leq b$, i.e., the relation $\preceq$ coincides with the usual order on rational numbers. But $a \circ b =$ least upper bound of a and b respect to $\preceq$, which is the same relation as $\leq$, so $a \circ b = \max(a, b)$.

The other possibility is that 0^* consists of all negative rational numbers and zero. In this case we have that $a \preceq b \Leftrightarrow a \geq b$, and $a \circ b = \min(a, b)$. □

Remark: The result does not hold on extensions of $\mathbb{Q}$, for instance on $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} \mid r, s \in \mathbb{Q}\}$ define the conjugate of an element $x = r + s\sqrt{2}$ as $\overline{x} = r - s\sqrt{2}$. Then the following binary operation verifies the hypotheses of the problem:

$$x \circ y = \overline{\max(\overline{x}, \overline{y})} \,,$$

and is different from $\min(x, y)$ and $\max(x, y)$.

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