Bijection between Two Sets. Let X be the set $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let P be the number of ordered pairs $(x, y) \in X^2$ with $x \neq y$ and let S be the number or 3-element subsets of X. We have $|P| = 8 \cdot 7 = 56$, and $|S| = \binom{8}{3} = 56$. Since they have the same number of elements then there is a bijection between P and S. Is there a bijection $f\colon P \to S$ such that for every two different elements x, y from X, x and y are in $f(x, y)$?

Answer. The answer is affirmative.

Consider the graph G whose vertices are $P \cup S$, and its edges join each element (x, y) from P to each 3-element subset of X containing x and y. The graph G is bipartite, and also k-regular with $k = 6$ since each vertex has exactly 6 neighbors, i.e., for each (x, y), x and y are contained in $\binom{6}{1}$ = six 3-element subsets of X, and for each 3-element subset $\{x, y, z\}$ of X there are $3 \cdot 2 = 6$ ordered pairs whose elements are in $\{x, y, z\}$.

Next, we use the following theorem:

Theorem 1. In any k-regular bipartite graph with equal partition sizes P and S, a perfect matching exists.[1]

Proof. We use Hall's Marriage Theorem which provides a necessary and sufficient condition for the existence of a perfect matching in bipartite graphs with equal parts. The Hall's condition is: For every subset A of P, the neighborhood $N(A)$ in S satisfies $|N(A)| \geq |A|$. This condition is in fact satisfied by any k-regular bipartite graph with equal partition sizes because, taking into account $E(A) \subseteq E(N(A))$, we have

$$k \cdot |A| = |E(A)| \leq |E(N(A))| = k \cdot |N(A)| .$$

hence $|A| \leq |N(A)|$, and the theorem is proved. □

Going back to the problem, we have a bipartite k-regular graph G, which by theorem 1 has a perfect matching M. Then, the desired bijection $f\colon P \to S$ can be obtained by mapping each ordered pair (x, y) in P to the element of S matched to (x, y) by M.

This completes the proof of the assertion.

□

Miguel A. Lerma - 9/24/2024

[1]A P-perfect (of P saturated) matching in a bypartite graph with parts P and S is a matching with disjoint edges that covers every vertex in P. If the two parts have equal size $|P| = |S|$, then a P-perfect matching is also an S-perfect matching, and it is not necessary to specify respect to a which part it is saturated.

Epilog. The result shown above is merely existential and does not produce any specific bijection with the required property. The following is an example of bijection from P to S that satisfies the desired property:

$$
\begin{array}{lllllll}
(1,2) & \mapsto & (1,2,4) & \quad & (1,3) & \mapsto & (1,3,5) \\
(1,4) & \mapsto & (1,4,7) & & (1,5) & \mapsto & (1,4,5) \\
(1,6) & \mapsto & (1,2,6) & & (1,7) & \mapsto & (1,5,7) \\
(1,8) & \mapsto & (1,7,8) & & (2,1) & \mapsto & (1,2,5) \\
(2,3) & \mapsto & (2,3,6) & & (2,4) & \mapsto & (2,4,6) \\
(2,5) & \mapsto & (2,3,5) & & (2,6) & \mapsto & (2,5,6) \\
(2,7) & \mapsto & (2,3,7) & & (2,8) & \mapsto & (1,2,8) \\
(3,1) & \mapsto & (1,2,3) & & (3,2) & \mapsto & (2,3,4) \\
(3,4) & \mapsto & (1,3,4) & & (3,5) & \mapsto & (3,5,6) \\
(3,6) & \mapsto & (1,3,6) & & (3,7) & \mapsto & (1,3,7) \\
(3,8) & \mapsto & (3,4,8) & & (4,1) & \mapsto & (1,4,6) \\
(4,2) & \mapsto & (2,4,5) & & (4,3) & \mapsto & (3,4,6) \\
(4,5) & \mapsto & (4,5,8) & & (4,6) & \mapsto & (4,6,7) \\
(4,7) & \mapsto & (4,7,8) & & (4,8) & \mapsto & (1,4,8) \\
(5,1) & \mapsto & (1,5,8) & & (5,2) & \mapsto & (2,5,7) \\
(5,3) & \mapsto & (3,4,5) & & (5,4) & \mapsto & (4,5,6) \\
(5,6) & \mapsto & (5,6,8) & & (5,7) & \mapsto & (5,6,7) \\
(5,8) & \mapsto & (2,5,8) & & (6,1) & \mapsto & (1,6,7) \\
(6,2) & \mapsto & (2,6,7) & & (6,3) & \mapsto & (3,6,8) \\
(6,4) & \mapsto & (4,6,8) & & (6,5) & \mapsto & (1,5,6) \\
(6,7) & \mapsto & (6,7,8) & & (6,8) & \mapsto & (2,6,8) \\
(7,1) & \mapsto & (1,2,7) & & (7,2) & \mapsto & (2,4,7) \\
(7,3) & \mapsto & (3,4,7) & & (7,4) & \mapsto & (4,5,7) \\
(7,5) & \mapsto & (3,5,7) & & (7,6) & \mapsto & (3,6,7) \\
(7,8) & \mapsto & (3,7,8) & & (8,1) & \mapsto & (1,3,8) \\
(8,2) & \mapsto & (2,3,8) & & (8,3) & \mapsto & (3,5,8) \\
(8,4) & \mapsto & (2,4,8) & & (8,5) & \mapsto & (5,7,8) \\
(8,6) & \mapsto & (1,6,8) & & (8,7) & \mapsto & (2,7,8)
\end{array}
$$

Added Remark. The result can be extended in the following way: Let n be a positive integer, and let $m = (n+1)!+n$.[2] If T is the set of ordered n-tuplas with distinct elements of $\{1, \ldots, m\}$, and S is the set of $(n+1)$-element subsets of $\{1, \ldots, m\}$, then there is a bijection $f\colon T \to S$ such that for each n-tupla $(x_1, \ldots, x_n) \in T$, $x_1, \ldots, x_n$ are in $f(x_1, \ldots, x_n)$. The proof is analogous to the one shown above, using the graph with vertices $T \cup S$, and edges joining each tupla $(x_1, \ldots, x_n)$ from T with each n-elements subset of $\{1, \ldots, m\}$ containing $x_1, \ldots, x_n$. The graph G is bipartite with $|T| = |S| = \binom{m}{n+1}$ and k-regular with $k = (n+1)!$, so the same argument used to solve the given proven can be used to prove this result. The problem posed covers the case $n = 2$.

[2] The sequence $(n+1)! + n = 1, 3, 8, 27, 124, 725, \ldots$ is A030495 in The On-Line Encyclopedia of Integer Sequences.