A Property of Representations of Integers as Combinations of Powers of Real Numbers

Problem. Let $\psi > 0$ be a real number and $m > 1$ an integer. Suppose that

$$\sum_{i=1}^{m} \psi^{a_i} = m$$

for nonzero integers $a_1, \ldots, a_m$. Show that every positive integer n can be written as

$$n = \sum_i c_i \psi^{k_i},$$

where the k_i are distinct integers, $c_i \in \{1, \ldots, m-1\}$ for all i, and $\sum_i c_i = n$.

Solution. The problem is equivalent to asserting that every positive integer n can be written as

$$n = \sum_{i=1}^{n} \psi^{k_i},$$

where no integer exponent appears m or more times.

If $\psi = 1$ the result is trivial, so we assume $\psi \neq 1$. Note that not all a_i can be equal, since that would imply $\psi = 1$. Without loss of generality we assume $\psi > 1$ and $a_1 \geq a_2 \geq \cdots \geq a_m$. Then $a_1 > 0$, since otherwise all $a_i < 0$, which would yield $\sum \psi^{a_i} < m$.

For $n > 0$, define

$$T_n = \Big\{(k_1, \ldots, k_n) \in \mathbb{Z}^n : n = \sum_{i=1}^{n} \psi^{k_i}\Big\}.$$

The tuple $(0, 0, \ldots, 0)$ always belongs to T_n, so $T_n \neq \varnothing$. We seek a tuple in T_n where no exponent occurs m or more times.

Replacement Step. Whenever a tuple contains m identical entries k, we may replace those m occurrences using

$$m\psi^k = \left(\sum_{i=1}^{m} \psi^{a_i}\right) \psi^k = \sum_{i=1}^{m} \psi^{k+a_i}.$$

The resulting tuple remains in T_n.

Lexicographic Argument. We order T_n lexicographically. Since $\psi^{k_i} \leq n$, all coordinates are bounded above by $\log n / \log \psi$, so T_n is finite and possesses a last element $\hat{t} = (\hat{k}_1, \ldots, \hat{k}_n)$ under this order.

Suppose some $\hat{k}$ appears m times in $\hat{t}$. Applying the replacement step to those m equal entries yields a new tuple $t' \in T_n$ that is lexicographically larger (since $a_1 > 0$). This contradicts the maximality of $\hat{t}$. Hence, in $\hat{t}$ no exponent appears m or more times.

Grouping terms with equal exponents gives

$$n = \sum_i c_i \psi^{k_i},$$

with $c_i \in \{1, \dots, m-1\}$ and $\sum_i c_i = n$, as required.

Remarks. The representation need not be unique. For example, with $\psi = \varphi = \frac{1+\sqrt{5}}{2}$ we have $\varphi^1 + \varphi^{-2} = 2$ ($m = 2$), and

$$3 = \varphi^2 + \varphi^{-3} + \varphi^{-4} = \varphi^1 + \varphi^0 + \varphi^{-2}.$$

This resembles the non-uniqueness of Zeckendorf-type representations.

Searching for Representations. Starting from $n = \sum_{i=1}^n \psi^0$, one can repeatedly apply the replacement step. Each replacement produces a lexicographically larger tuple, and since T_n is finite, the process must terminate after finitely many steps, yielding a valid representation.

Formally, any ascending sequence $t_1 \preceq_{\text{lex}} t_2 \preceq_{\text{lex}} \cdots$ in T_n must be finite, since $\mathbb{N}^n$ with the lexicographic order admits no infinite descending sequences. Thus, the iterative process halts in a tuple with no exponent repeated m times.

Alternative Solution. Define, for each n, the property

$P(n):$ "Given $r > 0$, the set of integer tuples $(k_1, \dots, k_n)$ such that $r = \displaystyle\sum_{i=1}^{n} \psi^{k_i}$ is finite."

We show by induction that $P(n)$ holds for all n. For $n = 1$ it is immediate. Assuming $P(n)$, note that for $r = \sum_{i=1}^{n+1} \psi^{k_i}$ we can bound k_1 via

$$\log\left(\frac{r}{n+1}\right)\Big/\log\psi \le k_1 < \log r / \log\psi,$$

so k_1 takes only finitely many integer values, and for each, there are finitely many continuations by the induction hypothesis.

Hence T_n is finite. Moreover, by AM–GM inequality,

$$1 = \frac{1}{m}\sum_{i=1}^{m} \psi^{a_i} > \left(\psi^{a_1 + \cdots + a_m}\right)^{1/m},$$

implying $\sum a_i < 0$. Each replacement step decreases the sum $\sum k_i$ of exponents, producing distinct tuples. Since T_n is finite, the process must terminate with the desired form.

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