VARIOUS TECHNIQUES

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(Last updated: April 14, 2003)

1. Definitions and Notations

1.1. Notations.

Integer Part, Fractional Part. The integer part of a real number x is $\lfloor x \rfloor$ = greatest integer less than or equal to x. The fractional part of a real number x is $\langle x \rangle = x - \lfloor x \rfloor$.

1.2. Some Definitions.

Bernoulli polynomials. The Bernoulli polynomials $\mathrm{B}_n(x)$ can be defined in various ways. The following are two of them:

(1) By a generating function:

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \mathrm{B}_n(x)\, \frac{t^n}{n!}$$

(2) By the following recursive formulas ($n \geq 1$):

(1) $\mathrm{B}_0(x) = 1\,,$

(2) $\mathrm{B}'_n(x) = n\, \mathrm{B}_{n-1}(x)\,,$

(3) $\displaystyle\int_0^1 \mathrm{B}_n(x)\, dx = 0\,.$

The first few Bernoulli polynomials are:

$$\begin{array}{ll} \mathrm{B}_0(x) = 1 & \mathrm{B}_1(x) = x - \frac{1}{2} \\ \mathrm{B}_2(x) = x^2 - x + \frac{1}{6} & B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{array}$$

The Bernoulli numbers are $B_n = \mathrm{B}_n(0)$.

2. Various Techniques

2.1. **Summation by Parts.** This is a formula that resembles integration by parts:

$$\sum_{k=m}^{n} a_k\, b_k = [a_n\, S_n - a_m\, S_{m-1}] + \sum_{k=m}^{n-1} (a_k - a_{k+1}) S_k\,,$$

where $S_k = \sum_{j=t}^{k} b_j$ (the lower limit is arbitrary).

Example: Prove that the following series converges for every x not an integer:

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n}\,.$$

Answer: We call

$$S_N = \sum_{n=0}^{N} e^{2\pi i n x} = \frac{e^{2\pi i (N+1) x} - 1}{e^{2\pi i x} - 1} \qquad (x \notin \mathbb{Z})\,,$$

hence

$$|S_N| \leq \frac{2}{|e^{2\pi i x} - 1|}\,.$$

Next using summation by parts:

$$\begin{aligned} \sum_{n=1}^{N} \frac{e^{2\pi i n x}}{n} &= \frac{1}{N} S_N - \frac{1}{1} S_0 + \sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1} \right) S_n \\ &= \frac{S_N}{N} - 1 + \sum_{n=1}^{N-1} \frac{S_n}{n(n+1)}\,. \end{aligned}$$

Letting $N \to \infty$ we get a series that converges absolutely by comparison with

$$\sum_{n=1}^{\infty} \frac{2/|e^{2\pi i x} - 1|}{n(n+1)} = \frac{2}{|e^{2\pi i x} - 1|} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \overset{\text{(telescopic)}}{=} \frac{2}{|e^{2\pi i x} - 1|}\,.$$

3. Summation Formulas

3.1. **The Euler-Maclaurin Summation Formula.** Let $f : [a,b] \to \mathbb{C}$ be q times differentiable, $\int_a^b |f^{(q)}(x)|\,dx < \infty$. Then for $1 \le m \le q$:

$$\begin{aligned}\sum_{a<n\le b} f(n) = \int_a^b f(x)\,dx \\ + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \left(\mathrm{B}_k(\langle b\rangle)\, f^{(k-1)}(b) - \mathrm{B}_k(\langle a\rangle)\, f^{(k-1)}(a)\right) \\ + \frac{(-1)^{m+1}}{m!} \int_a^b \mathrm{B}_m(\langle x\rangle)\, f^{(m)}(x)\,dx\,,\end{aligned}$$

where $B_k(x) = k$th Bernoulli polynomial.

Sum of Powers. As an example of application of the Euler-Maclaurin summation formula, we give the sum of the first m rth powers:

$$S(m,r) = \sum_{n=1}^{m} n^r = 1^r + 2^r + 3^r + \cdots + m^r\,.$$

Here $f(x) = x^r$, so $f^{(k)}(x) = r!x^{(r-k)}/(r-k)!$ for $k = 0, 1, \ldots, r$, $f^{(k)}(x) = 0$ for $k > r$, and

$$\begin{aligned}\sum_{0<n\le m} n^r = \int_0^m x^r\,dx \\ + \sum_{k=1}^{r+1} \frac{(-1)^k}{k!} \left(\mathrm{B}_k(\langle m\rangle)\, f^{(k-1)}(m) - \mathrm{B}_k(\langle 0\rangle)\, f^{(k-1)}(0)\right) \\ = \frac{m^{r+1}}{r+1} + \sum_{k=1}^{r+1} \frac{(-1)^k}{k!} B_k \frac{r!}{(r-k+1)!} m^{r-k+1} - \frac{B_{r+1}}{r+1} \\ = \frac{m^{r+1}}{r+1} + \frac{1}{r+1} \sum_{k=1}^{r+1} (-1)^k \tbinom{r+1}{k} B_k\, m^{r-k+1} - \frac{B_{r+1}}{r+1}\end{aligned}$$

Hence

$$S(m,r) = \sum_{n=1}^{m} n^r = \frac{1}{r+1} \left\{ \left(\sum_{k=0}^{r+1} (-1)^k \tbinom{r+1}{k} B_k\, m^{r-k+1} \right) - B_{r+1} \right\}$$

where B_k are the Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, $B_3 = 1/6$, etc.

For instance, for $r = 2$ we get:

$$\begin{aligned} S(m,2) &= \frac{1}{3}\left\{\left(\sum_{k=0}^{3}(-1)^k \binom{3}{k} B_k\, m^{3-k}\right) - B_3\right\} \\ &= \frac{1}{3}\left\{B_0\, m^3 - 3B_1\, m^2 + 3B_2\, m - B_3 - B_3\right\} \\ &= \frac{1}{3}\left\{m^3 + \frac{3}{2}m^2 + \frac{1}{2}m\right\} \\ &= \frac{2m^3 + 3m^2 + m}{6}\,. \end{aligned}$$

3.2. **The Poisson Summation Formula.** Here f represents a function $f : \mathbb{R} \to \mathbb{C}$. The Fourier transform of f is defined in the following way:

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x t}\, dx\,.$$

Periodic Version of a Function. The "periodic version" of f is defined as follows:

$$f_{per}(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(x+n)\,.$$

If f is absolutely integrable over $\mathbb{R}$, i.e., integrable and $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, then $f_{per}(x)$ exists for a.e.[1] x and is periodic: $f_{per}(x+1) = f_{per}(x)$.

Furthermore:

$$\widehat{f}(k) = \int_{0}^{1} f_{per}(x)\, e^{-2\pi i k x}\, dx\,.$$

Poisson Summation Formula. If f is absolutely integrable over $\mathbb{R}$, of bounded variation and normalized in the sense that for every x,

$$f(x) = \frac{1}{2}\lim_{h\to 0}\{f(x+h) + f(x-h)\}\,,$$

then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \lim_{T\to\infty}\sum_{k=-T}^{T} \widehat{f}(k)\, e^{2\pi i k x}.$$

[1]Every x but a set of measure zero.