SOME INEQUALITIES

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Introduction. These are a few useful inequalities. Most of them are presented in two versions: in sum form and in integral form. More generally they can be viewed as inequalities involving vectors, the sum version applies to vectors in $\mathbb{R}^n$ and the integral version applies to spaces of functions.

First a few notations and definitions.

Absolute value. The absolute value of x is represented $|x|$.

Norm. Boldface letters line $\mathbf{u}$ and $\mathbf{v}$ represent vectors. Their scalar product is represented $\mathbf{u}\cdot\mathbf{v}$. In $\mathbb{R}^n$ the scalar product of $\mathbf{u} = (a_1, \dots, a_n)$ and $\mathbf{v} = (b_1, \dots, b_n)$ is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} a_i b_i \,.$$

For functions $f, g : [a, b] \to \mathbb{R}$ their scalar product is

$$\int_a^b f(x) g(x) \, dx \,.$$

The p-norm of $\mathbf{u}$ is represented $\|\mathbf{u}\|_p$. If $\mathbf{u} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, its p-norm is:

$$\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} .$$

For functions $f : [a, b] \to \mathbb{R}$ the p-norm is defined:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} .$$

For $p = 2$ the norm is called Euclidean.

Convexity. A function $f : I \to \mathbb{R}$ (I an interval) is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for every $x, y \in I$, $0 \le \lambda \le 1$. Graphically, the condition is that for $x < t < y$ the point $(t, f(t))$ should lie below or on the line connecting the points $(x, f(x))$ and $(y, f(y))$.

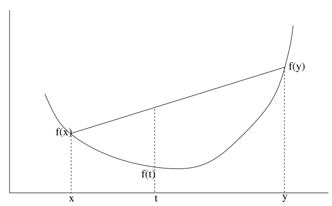

FIGURE 1. Convex function.

INEQUALITIES

1. **Arithmetic-Geometric Mean Inequality.** (Consequence of convexity of e^x and Jensen's inequality.) The geometric mean of positive numbers is not greater than their arithmetic mean, i.e., if $a_1, a_2, \dots, a_n > 0$, then

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n} a_i \,.$$

Equality happens only for $a_1 = \cdots = a_n$. (See also the power means inequality.)

2. **Arithmetic-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their arithmetic mean, i.e., if $a_1, a_2, \dots, a_n > 0$, then

$$\frac{n}{\sum_{i=1}^{n} 1/a_i} \le \frac{1}{n}\sum_{i=1}^{n} a_i \,.$$

Equality happens only for $a_1 = \cdots = a_n$.

This is a particular case of the Power Means Inequality.

3. **Cauchy.** (Hölder for $p = q = 2$.)

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \,.$$

$$\left|\sum_{i=1}^n a_i b_i\right|^2 \le \left|\sum_{i=1}^n a_i^2\right| \left|\sum_{i=1}^n b_i^2\right| .$$

$$\left|\int_a^b f(x) g(x)\, dx\right|^2 \le \left(\int_a^b |f(x)|^2\, dx\right) \left(\int_a^b |g(x)|^2\, dx\right) .$$

4. **Chebyshev.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of real numbers which are monotonic in the same direction (we have $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$, or we could reverse all inequalities.) Then

$$\frac{1}{n} \sum_{i=1}^n a_i b_i \ge \left(\frac{1}{n} \sum_{i=1}^n a_i\right) \left(\frac{1}{n} \sum_{i=1}^n b_i\right) .$$

Note that LHS $-$ RHS $= \dfrac{1}{2n^2} \displaystyle\sum_{i,j} (a_i - a_j)(b_i - b_j) \ge 0$.

5. **Geometric-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their geometric mean, i.e., if $a_1, a_2, \ldots, a_n > 0$, then

$$\frac{n}{\sum_{i=1}^n 1/a_i} \le \left(\prod_{i=1}^n a_i\right)^{1/n} .$$

Equality happens only for $a_1 = \cdots = a_n$.

This is a particular case of the Power Means Inequality.

6. **Hölder.** If $p > 1$ and $1/p + 1/q = 1$ then

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|_p \|\mathbf{v}\|_q \,.$$

$$\left|\sum_{i=1}^n a_i b_i\right| \le \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |b_i|^p\right)^{1/q} .$$

$$\left|\int_a^b f(x) g(x)\, dx\right| \le \left(\int_a^b |f(x)|^p\, dx\right)^{1/p} \left(\int_a^b |g(x)|^q\, dx\right)^{1/q} .$$

7. **Jensen.** If φ is convex on (a,b), $x_1, x_2, \dots, x_n \in (a,b)$, $\lambda_i \ge 0$ $(i = 1, 2, \dots, n)$, $\sum_{i=1}^n \lambda_i = 1$, then

$$\varphi\Big(\sum_{i=1}^n \lambda_i x_i\Big) \le \sum_{i=1}^n \lambda_i \varphi(x_i)\,.$$

8. **MacLaurin's Inequalities.** Let e_k be the kth degree elementary symmetric polynomial in n variables:

$$e_k(x_1, x_2, \dots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}\,.$$

Given positive numbers $a_1, a_2, \dots, a_n$, let $S_k = e_k(a_1, a_2, \dots, a_n)/\binom{n}{k}$ be the averages of the elementary symmetric sums of the a_i. Then

$$S_1 \ge \sqrt{S_2} \ge \sqrt[3]{S_3} \ge \cdots \ge \sqrt[n]{S_n}\,,$$

with equality if and only if all the a_i are equal. (See [3].) (See also Newton's Inequalities.)

9. **Minkowski.** If $p > 1$ then

$$\|\mathbf{u} + \mathbf{v}\|_p \le \|\mathbf{u}\|_p + \|\mathbf{v}\|_p\,,$$

$$\Big(\sum_{i=1}^n |a_i + b_i|^p\Big)^{1/p} \le \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^n |b_i|^p\Big)^{1/p},$$

$$\left(\int_a^b |f(x) + g(x)|^p\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}.$$

Equality holds iff $\mathbf{u}$ and $\mathbf{v}$ are proportional.

10. **Muirhead's Inequality.** Given real numbers $a_1 \ge \cdots \ge a_n$, and $b_1 \ge \cdots \ge b_n$, assume that $\sum_{i=1}^k a_i \le \sum_{i=1}^k b_i$ for $i = 1, \dots, n-1$, and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$. Then for any nonnegative real numbers $x_1, \dots, x_n$, we have

$$\sum_\sigma x_{\sigma_1}^{a_1} \cdots x_{\sigma_n}^{a_n} \le \sum_\sigma x_{\sigma_1}^{b_1} \cdots x_{\sigma_n}^{b_n}\,,$$

where the sums extend over all permutations σ of $\{1, \dots, n\}$ (see theorem 2.18 in [1].)

11. **Newton's Inequalities.** Let e_k be the kth degree elementary symmetric polynomial in n variables:

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k} \,.$$

Given positive numbers $a_1, a_2, \ldots, a_n$, let $S_k = e_k(a_1, a_2, \ldots, a_n)/\binom{n}{k}$ be the averages of the elementary symmetric sums of the a_i for $k \ge 1$, and $S_0 = 1$. Then (for $k = 1, 2, \ldots, n-1$):

$$S_{k-1} S_{k+1} \le S_k^2 \,,$$

with equality if and only if all the a_i are equal. (See also MacLaurin's Inequalities).

12. **Norm Monotonicity.** If $a_i > 0$ ($i = 1, 2, \ldots, n$), $s > t > 0$, then

$$\left(\sum_{i=1}^{n} a_i^s\right)^{1/s} \le \left(\sum_{i=1}^{n} a_i^t\right)^{1/t} ,$$

i.e., if $s > t > 0$, then $\|\mathbf{u}\|_s \le \|\mathbf{u}\|_t$.

13. **Power Means Inequality.** Let r be a non-zero real number. We define the r-mean or rth power mean of non-negative numbers $a_1, \ldots, a_n$ as follows:

$$M^r(a_1, \ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^r\right)^{1/r} .$$

If $r < 0$, and $a_k = 0$ for some k, we define $M^r(a_1, \ldots, a_n) = 0$.

The ordinary arithmetic mean is M^1, M^2 is the quadratic mean, M^{-1} is the harmonic mean. Furthermore we define the 0-mean to be equal to the geometric mean:

$$M^0(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i\right)^{1/n} .$$

Then for any real numbers r, s such that $r < s$, the following inequality holds:

$$M^r(a_1, \ldots, a_n) \le M^s(a_1, \ldots, a_n) \,.$$

Equality holds if and only if $a_1 = \cdots = a_n$, or $s \le 0$ and $a_k = 0$ for some k. (See weighted power means inequality).

14. **Power Means Sub/Superadditivity.** We use the definition of r-mean given in subsection 13. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be non-negative real numbers.

(1) If $r > 1$, then the r-mean is *subadditive*, i.e.:

$$M^r(a_1 + b_1, \dots, a_n + b_n) \leq M^r(a_1, \dots, a_n) + M^r(b_1, \dots, b_n)\,.$$

(2) If $r < 1$, then the r-mean is *superadditive*, i.e.:

$$M^r(a_1 + b_1, \dots, a_n + b_n) \geq M^r(a_1, \dots, a_n) + M^r(b_1, \dots, b_n)\,.$$

Equality holds if and only if $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are proportional, or $r \leq 0$ and $a_k = b_k = 0$ for some k.

15. **Radon's Inequality.** For real numbers $p > 0$, $x_1, \dots, x_n \geq 0$, $a_1, \dots, a_n > 0$, the following inequality holds:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^n x_k)^{p+1}}{(\sum_{k=1}^n a_k)^p}\,.$$

Remark: Radon's Inequality follows from Hölder's $|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|_{p+1}\|\mathbf{v}\|_q$, with $\mathbf{u} = (x_1/a_1^{1/q}, \dots, x_n/a_n^{1/q})$, $\mathbf{v} = (a_1^{1/q}, \dots, a_n^{1/q})$, $\frac{1}{p+1} + \frac{1}{q} = 1$.

16. **Rearrangement Inequality.** For every choice of real numbers $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$, and any permutation $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ of $x_1, \dots, x_n$, we have

$$x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots + x_n y_n\,.$$

If the numbers are different, e.g., $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, then the lower bound is attained only for the permutation which reverses the order, i.e. $\sigma(i) = n - i + 1$, and the upper bound is attained only for the identity, i.e. $\sigma(i) = i$, for $i = 1, \dots, n$.

17. **Schur.** If x, y, x are positive real numbers and k is a real number such that $k \geq 1$, then

$$x^k(x-y)(x-z) + y^k(y-x)(y-z) + z^k(z-x)(z-y) \geq 0\,.$$

For $k = 1$ the inequality becomes

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)\,.$$

18. **Schwarz.** (Hölder with $p = q = 2$.)

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \,,$$

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right),$$

$$\left|\int_a^b f(x)g(x)\,dx\right|^2 \leq \left(\int_a^b |f(x)|^2\,dx\right)\left(\int_a^b |g(x)|^2\,dx\right).$$

19. **Strong Mixing Variables Method.** We use the definition of r-mean given in subsection 13. Let $F : I \subset \mathbb{R}^n \to \mathbb{R}$ be a symmetric, continuous function satisfying the following: for all $(x_1, x_2, \ldots, x_n) \in I$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, $F(x_1, x_2, \ldots, x_n) \geq F(t, x_2, \ldots, x_{n-1}, t)$, where $t = M^r(x_1, x_n)$. Then:

$$F(x_1, x_2, \ldots, x_n) \geq F(x, x, \ldots, x)\,,$$

where $x = M^r(x_1, x_2, \ldots, x_n)$.

An analogous result holds replacing $\geq$ with $\leq$.

20. **Weighted Power Means Inequality.** Let $w_1, \ldots, w_n$ be positive real numbers such that $w_1 + \cdots + w_n = 1$. Let r be a non-zero real number. We define the rth weighted power mean of non-negative numbers $a_1, \ldots, a_n$ as follows:

$$M_w^r(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} w_i a_i^r\right)^{1/r}.$$

As $r \to 0$ the rth weighted power mean tends to:

$$M_w^0(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i^{w_i}\right).$$

which we call 0th weighted power mean. If $w_i = 1/n$ we get the ordinary rth power means.

Then for any real numbers r, s such that $r < s$, the following inequality holds:

$$M_w^r(a_1, \ldots, a_n) \leq M_w^s(a_1, \ldots, a_n)\,.$$

(If $r, s \neq 0$ note convexity of $x^{s/r}$ and recall Jensen's inequality.)

1. Various Results

1. **Convex Function Superadditivity.** If $f : [0, b) \to \mathbb{R}$ is convex, and $f(0) = 0$, then f is superadditive in $[0, b)$, i.e., if $x, y, x+y \in [0, b)$, then $f(x+y) \geq f(x) + f(y)$.

Proof: If $0 \leq \lambda \leq 1$, then by convexity:

$$f(\lambda x) = f(\lambda x + (1-\lambda)0) \leq \lambda f(x) + (1-\lambda) f(0) = \lambda f(x)\,,$$

hence

$$f(x) = f\left(\frac{x}{x+y}\,(x+y)\right) \leq \frac{x}{x+y} f(x+y)$$

$$f(y) = f\left(\frac{y}{x+y}\,(x+y)\right) \leq \frac{y}{x+y} f(x+y)\,,$$

and adding both inequalities: $f(x) + f(y) \leq f(x+y)$. □

References

[1] G. Hardy, J. E. Littlewood and G. Pólya. *Inequalities*, Second Edition. Cambridge University Press, 1952.

[2] J. Michael Steele. *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge University Press, 2004.

[3] Thomas Foregger, Andrei Ismail, Pedro Sanchez. *MacLaurin's inequality* (version 4). PlanetMath.org. Freely available at `http://planetmath.org/MacLaurinsInequality.html`.