GENERATING FUNCTIONS

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1. Generating Functions

1. **Definition.** Given a sequence $a_0, a_1, a_2, \dots$, the function

$$f(x) = a_0 + a_1\,x + a_2\,x^2 + \cdots = \sum_{n=0}^{\infty} a_n\,x^n$$

is called the *generating function* for the given sequence.

Examples:

(1) $(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n$ is the generating function for

$$\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \dots, \binom{m}{m}, 0, 0, 0, \dots .$$

(2) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$ is the generating function for the constant sequence $a_n = 1$.

(3) The generating function of a geometric sequence $a_n = A\,r^n$ is

$$\frac{A}{1-r\,x} = A + A\,r\,x + A\,r^2\,x^2 + A\,r^3\,x^3 + \dots$$

(4) If $f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots$ is the generating function for $a_0, a_1, a_2, \dots$, then the following is the generating function for the sequence of partial sums $\sum_{k=0}^{n} a_k$:

$$\begin{aligned}\frac{f(x)}{1-x} &= f(x)(1 + x + x^2 + x^3 + \cdots)\\ &= (a_0 + a_1\,x + a_2\,x^2 + \cdots)(1 + x + x^2 + x^3 + \cdots)\\ &= a_0 + (a_0 + a_1)\,x + (a_0 + a_1 + a_2)\,x^2 + \cdots\end{aligned}$$

(5) *Partitions of Integers*. Let $p_M(n)$ be the number of ways the positive integer n can be written as a sum of positive integers

not exceeding M. For instance $p_3(5) = 5$ because $5 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Then

$$\prod_{k=1}^{M} \frac{1}{1 - x^k} = \sum_{n=0}^{\infty} p_M(n)\, x^k \,.$$

Example: Use generating functions to find the sum of the terms of a geometric sequence: $s_n = 1 + r + r^2 + \cdots + r^n$ $(r \neq 1)$. *Answer*: We have

$$\frac{1}{1 - rx} = \sum_{n=0}^{\infty} r^n x^n \,,$$

hence

$$\frac{1}{(1 - x)(1 - rx)} = \sum_{n=0}^{\infty} s_n x^n \,.$$

Next, we expand the left hand side into partial fractions:

$$\begin{aligned}
\frac{1}{(1 - x)(1 - rx)} &= \frac{1}{1 - r} \left(\frac{1}{1 - x} - \frac{r}{1 - rx} \right) \\
&= \frac{1}{r - 1} \left(\sum_{n=0}^{\infty} r^{n+1} x^n - \sum_{n=0}^{\infty} x^n \right) \\
&= \sum_{n=0}^{\infty} \frac{r^{n+1} - 1}{r - 1}\, x^n \,,
\end{aligned}$$

hence

$$s_n = \frac{r^{n+1} - 1}{r - 1} \,.$$

2. **Generating Function of a Linear Homogeneous Recurrence.** Given a kth-order linear homogeneous recurrence relation:

$$x_n + c_1\, x_{n-1} + c_2\, x_{n-2} + \cdots + c_k\, x_{n-k} = 0$$

(note that the leading coefficient is 1), its characteristic polynomial is

$$Q(r) = r^k + c_1\, r^{n-1} + c_2\, r^{n-2} + \cdots + c_k \,.$$

Let $q(x) = x^k\, Q(x^{-1})$ be the polynomial[1]

$$q(x) = x^k\, Q(x^{-1}) = 1 + c_1\, x + c_2\, x^2 + \cdots c_k\, x^k \,.$$

[1]The polynomial $q(x)$ is called *reciprocal* of $Q(x)$.

Then the solution of the recurrence has the following generating function:

$$f(x) = \frac{p(x)}{q(x)}\,,$$

where $p(x)$ is some polynomial of degree less than r which depends on the initial conditions for the recurrence.

Example: We use the method of generating functions to obtain the close-form formula for the Fibonacci sequence. The Fibonacci sequence F_n has the following characteristic polynomial:

$$P(x) = r^2 - r - 1\,,$$

so $q(x) = 1 - x - x^2$. Hence,

$$\frac{a + b\,x}{1 - x - x^2} = F_0 + F_1\,x + F_2\,x^2 + \cdots\,.$$

We determine the numerator of the generating function by writing

$$\begin{aligned} a + b\,x &= (1 - x - x^2)\,(F_0 + F_1\,x + F_2\,x^2 + \cdots) \\ &= F_0 + \{F_1 - F_0\}\,x + \{F_2 - F_1 - F_0\}\,x^2 + \dots\,, \end{aligned}$$

so that $a = F_0 = 0$, $b = F_1 - F_0 = 1$, hence:

$$\frac{x}{1 - x - x^2} = F_0 + F_1\,x + F_2\,x^2 + \cdots\,.$$

We can factor the denominator like this

$$1 - x - x^2 = (1 - \phi\,x)\,(1 + \phi^{-1}\,x)\,,$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}\,; \qquad -\phi^{-1} = \frac{1 - \sqrt{5}}{2}\,.$$

The generating function can be expressed as a sum of partial fractions in the following way:

$$\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}}\left\{\frac{1}{1 - \phi\,x} - \frac{1}{1 + \phi^{-1}\,x}\right\}.$$

Since in general $1/(1 - a\,x)$ is the generating function for the sequence a^n, we get that

$$\begin{aligned} \frac{x}{1 - x - x^2} &= \frac{1}{\sqrt{5}}\left\{\sum_{n=0}^{\infty} \phi^n\,x^n - \sum_{n=0}^{\infty} (-\phi^{-1})^n\,x^n\right\} \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left\{\phi^n - (-\phi)^{-n}\right\}\,x^n\,. \end{aligned}$$

That must be equal to

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n \, x^n \,,$$

so we get:

$$F_n = \frac{1}{\sqrt{5}} \left\{ \phi^n - (-\phi)^{-n} \right\} \,.$$