A Property of Representations of Integers as Combinations of Powers of Real Numbers

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Abstract

Let $\psi > 0$ and integers $a_1, \dots, a_m \neq 0$ satisfy

$$\sum_{i=1}^{m} \psi^{a_i} = m. \tag{1}$$

We prove that every positive integer n admits a finite expansion

$$n = \sum_{j=1}^{t} c_j, \psi^{k_j}, \qquad k_1, \dots, k_t \in \mathbb{Z} \text{ pairwise distinct}, \quad c_j \in 1, \dots, m-1, \quad \sum_{j=1}^{t} c_j = n. \tag{2}$$

We provide two elementary proofs (a lexicographic maximal-element argument and a terminating carry process), identify a canonical normal form, and discuss connections with abelian chip–firing networks and β–expansions. A practical search procedure and examples (including the golden ratio) are included.

1 Introduction

Dresden and Liu (CMJ Problem 1276) [1] and the solution by the Eagle Problem Solvers [2] considered the special case $m = 2$ with

$$\psi^a + \psi^b = 2 \quad (a \neq b), \tag{3}$$

showing that every $n \in \mathbb{N}$ is a sum of n distinct powers of ψ. We generalize this to arbitrary $m > 1$ under (1) with $a_i \neq 0$ (the a_i not necessarily distinct). Throughout we may assume $\psi > 1$; if $\psi < 1$ we replace ψ by $1/\psi$ and (a_i) by $(-a_i)$, preserving (1).

2 Sketch of proof

The general idea of the proof consists of starting with an expression of the form

$$n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = \sum_{i=1}^{n} \psi^0, \tag{4}$$

which may contain m or more repetitions of the same power of ψ, and then repeatedly applying a rewriting rule of the form $m\psi^k = \sum_{i=1}^{m} \psi^{k+a_i}$. The argument shows that this process always terminates and leads to a stable configuration in which no exponent appears m or more times, yielding the desired canonical form. We borrow terminology from related works such as [3, 4, 5], where similar rewriting procedures are described in the contexts of abelian networks and β–expansions.

3 A carry rule and configurations

For any $k \in \mathbb{Z}$, multiplying (1) by ψ^k yields the *carry rule*

$$m\psi^k = \sum_{i=1}^{m} \psi^{k+a_i}. \tag{5}$$

A *configuration* is a finitely supported function $x\colon \mathbb{Z} \to \mathbb{N}$; interpret $x(k)$ as the coefficient (or number of chips) at exponent k. Define its *value* and *chip count* by

$$\mathrm{Val}(x) := \sum_{k\in\mathbb{Z}} x(k), \psi^k, \qquad |x| := \sum_{k\in\mathbb{Z}} x(k). \tag{6}$$

A *topple at k* replaces m copies of ψ^k by one copy of each ψ^{k+a_i}:

$$x'(k) = x(k) - m, \qquad x'(k + a_i) = x(k + a_i) + 1 \ (i = 1, \ldots, m), \tag{7}$$

leaving other coordinates unchanged. By (5), toppling preserves value and chip count:

$$\mathrm{Val}(x') = \mathrm{Val}(x), \qquad |x'| = |x|. \tag{8}$$

A configuration is *stable* if $x(k) \le m - 1$ for all k.

4 Main result

Theorem 1. *Assume* (1) *with $a_i \neq 0$. For every $n \in \mathbb{N}$ there exists a stable configuration x with $\mathrm{Val}(x) = n$ and $|x| = n$. Equivalently, n admits a representation* (2).

Proof via lexicographic maximal element

Fix $n > 0$ and consider the nonempty set

$$T_n := \big\{(k_1, \ldots, k_n) \in \mathbb{Z}^n \, : \, n = \sum_{i=1}^{n} \psi^{k_i}\big\}. \tag{9}$$

Step 1: Upper bounds. From $\psi^{k_i} \le n$ we get $k_i \le K := \lfloor \log n / \log \psi \rfloor$ for all i.

Step 2: Lexicographic maximal element exists. Order T_n lexicographically. Because each coordinate is bounded above and integers are well-ordered below, there is a maximal element $\hat{t} = (\hat{k}_1, \dots, \hat{k}_n)$: choose $\hat{k}_1$ to be the maximal possible first coordinate among tuples in T_n, then $\hat{k}_2$ maximal among those with first coordinate $\hat{k}_1$, and so on.

Step 3: No m-fold repetition in $\hat{t}$. Suppose some value k appears at least m times in $\hat{t}$. Apply one carry (5) to m occurrences of k, replacing them by $k + a_1, \dots, k + a_m$. Since at least one $a_i > 0$ (else (1) fails for $\psi > 1$), the resulting tuple is lexicographically larger than $\hat{t}$, contradiction. Therefore no exponent occurs m times in $\hat{t}$.

Step 4: Grouping identical exponents. Writing the multiset of entries of $\hat{t}$ as distinct exponents k_j with multiplicities $c_j \in 1, \dots, m-1$, we obtain (2). Since $|x| = n$ for the associated configuration, $\sum_j c_j = n$. □

5 Alternative proof: termination of the carry process

We show that repeatedly applying the carry (5) to any tuple in T_n with an m-fold repetition must halt at a tuple with all multiplicities $\leq m - 1$.

Proposition 1 (Finiteness of T_n)**.** *For each $n \in \mathbb{N}$ and $r > 0$, the equation $r = \sum_{i=1}^{n} \psi^{k_i}$ has only finitely many integer solutions $(k_1, \dots, k_n)$. In particular, T_n is finite.*

Proof. By induction on n. For $n = 1$, $r = \psi^{k_1}$ has at most one solution $k_1 = \log r / \log \psi$. Assume true for n. For $n + 1$, arrange $k_1 \geq \dots \geq k_{n+1}$. Then

$$\frac{\log\big(r/(n+1)\big)}{\log \psi} \ \leq \ k_1 \ < \ \frac{\log r}{\log \psi}, \tag{10}$$

so k_1 takes finitely many values; for each such k_1, set $r' = r - \psi^{k_1} > 0$ and apply the induction hypothesis to $r' = \sum_{i=2}^{n+1} \psi^{k_i}$. □

Lemma 1 (Strict descent of the sum of exponents)**.** *Assume additionally $\psi > 1$ and (1). Then $\sum_{i=1}^{m} a_i < 0$. Consequently, a single carry applied to m equal exponents strictly decreases the total sum of exponents of the tuple.*

Proof. By AM–GM,

$$1 = \frac{1}{m} \sum_{i=1}^{m} \psi^{a_i} \ > \ \big(\psi^{a_1} \cdots \psi^{a_m}\big)^{1/m} = \psi^{(a_1 + \dots + a_m)/m}, \tag{11}$$

where the inequality is strict since the ψ^{a_i} are not all equal (the a_i are not all the same because $a_i \neq 0$ and (1) with all a_i equal would force $\psi = 1$). As $\psi > 1$, (11) implies $a_1 + \dots + a_m < 0$. Replacing m occurrences of some k by $k + a_1, \dots, k + a_m$ changes the sum of entries by $\sum_i a_i < 0$. □

Proposition 2 (Termination)**.** *Starting from any tuple in T_n, repeated application of carries to any position with multiplicity $\geq m$ terminates after finitely many steps at a tuple in which all multiplicities are $\leq m - 1$.*

Proof. By Proposition 1, T_n is finite, so no infinite sequence of distinct tuples is possible. By Lemma 1, each carry strictly decreases the sum of entries, hence two consecutive tuples are distinct. Therefore the process halts, and the terminal tuple has no site with multiplicity $\geq m$ (otherwise we could carry once more). □

Combining Propositions 1 and 2 yields Theorem 1.

6 Canonical form and uniqueness

Among all tuples in T_n, the lexicographically maximal one $\hat{t}$ (constructed above) is *canonical*. Any sequence of carries starting from $(0, \ldots, 0)$ must terminate, and the terminal tuple's multiset of exponents is independent of the order of legal carries (the abelian/commutative nature of (7)). Grouping equal exponents gives the unique normalized expansion with coefficients in $0, \ldots, m-1$; deleting zero coefficients yields (2).

7 Practical search (algorithm)

Start with the length-n tuple $(0, \ldots, 0) \in T_n$. While some value appears $\geq m$ times, apply one carry (5) to m of those occurrences. By Proposition 2 this halts in finitely many steps with all multiplicities $\leq m-1$. Group identical exponents to obtain coefficients $c_j \in 1, \ldots, m-1$ summing to n.

8 Examples

Example 1 (Golden ratio)**.** Let $\phi = (1+\sqrt{5})/2$. Since $\phi^2 = \phi + 1$, we have $\phi^{-2} = 2 - \phi$ and hence

$$\phi^1 + \phi^{-2} = 2 \qquad (m = 2,\ a_1 = 1,\ a_2 = -2). \tag{12}$$

The carry is $2\phi^k \to \phi^{k+1} + \phi^{k-2}$. For $n = 3$,

$$3\phi^0 \ = \ (2\phi^0) + \phi^0 \ \xrightarrow{(12)} \ \phi^1 + \phi^{-2} + \phi^0 \ = \ \boxed{\phi^1 + \phi^0 + \phi^{-2}}, \tag{13}$$

which is the canonical form. Another valid (noncanonical) triple is $\phi^2 + \phi^{-3} + \phi^{-4}$.

Example 2 (A ternary case)**.** Take $\psi = 2$ and $(a_1, a_2, a_3) = (1, -1, -1)$. Then

$$2^1 + 2^{-1} + 2^{-1} = 2 + \tfrac{1}{2} + \tfrac{1}{2} = 3,$$

so (1) holds with $m = 3$ and all $a_i \neq 0$. The carry rule is

$$3\,2^k \ \longrightarrow \ 2^{k+1} + 2 \cdot 2^{k-1}.$$

Starting from $(0, \ldots, 0) \in T_n$ and carrying whenever a site has multiplicity ≥ 3 yields a stable representation of n with digits in $\{1, 2\}$ and distinct exponents.

Example 3 (Ternary case with distinct exponents)**.** Let $(a_1, a_2, a_3) = (1, -1, -2)$ and solve $\psi^1 + \psi^{-1} + \psi^{-2} = 3$ for $\psi > 1$. Multiplying by ψ^2 gives $\psi^3 - 3\psi^2 + \psi + 1 = (\psi - 1)(\psi^2 - 2\psi - 1) = 0$, so $\psi = 1 + \sqrt{2} > 1$ works. Thus

$$\psi^1 + \psi^{-1} + \psi^{-2} = 3 \qquad (\psi = 1 + \sqrt{2}).$$

The carry rule is

$$3\,\psi^k \;\longrightarrow\; \psi^{k+1} + \psi^{k-1} + \psi^{k-2},$$

with all three shifts distinct and nonzero. For instance,

$$3\psi^0 \to \psi^1 + \psi^{-1} + \psi^{-2}, \quad 4\psi^0 \to \psi^1 + \psi^{-1} + \psi^{-2} + \psi^0, \quad 5\psi^0 \to \psi^1 + \psi^{-1} + \psi^{-2} + 2\psi^0.$$

Example 4 (Representation of 7 for $\psi = 1 + \sqrt{2}$)**.** For the ternary case with $(a_1, a_2, a_3) = (1, -1, -2)$ and $\psi = 1 + \sqrt{2}$, the relation $\psi^1 + \psi^{-1} + \psi^{-2} = 3$ yields the carry rule

$$3\,\psi^k \;\longrightarrow\; \psi^{k+1} + \psi^{k-1} + \psi^{k-2}.$$

Step 1. Starting configuration. For $n = 7$ we begin with $7\psi^0 = (3\psi^0) + (3\psi^0) + \psi^0$ and apply the carry rule to each block of three equal powers:

$$7\psi^0 \;\longrightarrow\; (\psi^1 + \psi^{-1} + \psi^{-2}) + (\psi^1 + \psi^{-1} + \psi^{-2}) + \psi^0.$$

Collecting equal exponents gives

$$x(1) = 2, \qquad x(0) = 1, \qquad x(-1) = 2, \qquad x(-2) = 2.$$

Step 2. Stabilization. All coefficients are $\leq m - 1 = 2$, so the configuration is already stable. Hence the canonical representation is

$$7 = 2\psi^1 + \psi^0 + 2\psi^{-1} + 2\psi^{-2}, \qquad \psi = 1 + \sqrt{2}.$$

9 Remarks and connections

Remark 1 (Zero exponents)**.** If some $a_i = 0$, then removing all zero exponents reduces (1) to $\sum_{a_i \neq 0} \psi^{a_i} = m - r$, where r is the count of zeros. The cases $r \in m, m-1$ are trivial; for $0 < r \leq m - 2$ the theorem follows from the case $m' = m - r$.

Remark 2 (Beta-expansions)**.** Let $Q(T) = \sum_{i=1}^{m} T^{a_i} - m$. Then $Q(\psi) = 0$ is a finite relation for 1 in base ψ. The carry rule (5) corresponds to digit normalization in β–expansions with a finite "word" for 1; the chip count $|x|$ being invariant explains the constraint $\sum c_j = n$.

Remark 3 (Abelian networks)**.** Toppling operators commute and preserve (6); the existence and uniqueness of the stabilized state parallels the confluence/least-action principles in abelian chip–firing networks.

10 Open Questions

The results presented here guarantee the existence and uniqueness of a canonical representation for each integer n under the relation $\sum_{i=1}^{m} \psi^{a_i} = m$. Several natural questions remain open:

1. **Counting representations.** Beyond the canonical (stabilized) form, an integer n may admit other valid representations of the form $n = \sum_i c_i \psi^{k_i}$ with $c_i \in \{1, \ldots, m-1\}$. Describing or bounding this number as a function of n is an open combinatorial problem.

2. **Structure of noncanonical representations.** When multiple expansions exist, can one classify them via local transformations (inverse carries) or by a graph structure relating equivalent configurations?

3. **Asymptotic and statistical questions.** For large n, how are the exponents k_i distributed in the canonical form? Does their range grow logarithmically, linearly, or in some other predictable way?

4. **Computational aspects.** What is the complexity of computing the canonical representation of n? Can efficient algorithms or closed-form recurrences be obtained for specific values of ψ?

These directions connect the present work with combinatorial number theory, dynamical systems, and the study of β–expansions and abelian networks, offering a broad range of possibilities for further exploration.

11 References

References

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