Undecidability of Infiniteness and Intersections of Recursively Enumerable Sets

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1 Preliminaries

A set $A \subseteq \mathbb{N}$ is *recursively enumerable* (r.e.) if there exists a Turing machine that enumerates exactly the elements of A. Equivalently, membership in A is semi-decidable.

We write W_e for the r.e. set enumerated by the Turing machine with index e. Let

$$K = \{e : M_e(e) \text{ halts}\}$$

denote the diagonal halting set, which is well known to be undecidable.

We recall that the class of r.e. sets is closed under finite union and intersection, but not under complement.

2 Infiniteness of r.e. Sets

Theorem 1. *There is no algorithm that decides, given an index e, whether the r.e. set W_e is infinite.*

Proof. We reduce the halting problem to infiniteness.

For each e, construct an r.e. set S_e as follows. At stage $n = 0, 1, 2, \ldots$, simulate $M_e(e)$ for n steps. If the simulation halts within those n steps, enumerate n; otherwise enumerate nothing.

If $M_e(e)$ never halts, then $S_e = \varnothing$, which is finite. If $M_e(e)$ halts, then all sufficiently large n are enumerated, so S_e is infinite.

By the *s-m-n* theorem there is a computable function f with

$$W_{f(e)} = S_e,$$

so

$$e \in K \iff W_{f(e)} \text{ is infinite.}$$

Thus deciding infiniteness would decide K, a contradiction. □

3 Intersection with a Fixed Infinite r.e. Set

Theorem 2. *Let $A \subseteq \mathbb{N}$ be an infinite r.e. set. There is no algorithm that decides, given an index e, whether*

$$|W_e \cap A| = \infty.$$

Proof. Since A is infinite and r.e., fix a computable enumeration of distinct elements

$$A = \{a_0, a_1, a_2, \dots\}.$$

For each e, construct an r.e. set $S_e \subseteq A$ as follows. Simulate $M_e(e)$. If it never halts, enumerate only a_0. If it halts, enumerate $a_0, a_1, a_2, \dots$.

Then

$$|S_e \cap A| = \infty \iff M_e(e) \text{ halts.}$$

As before, this yields a many-one reduction from K, proving undecidability. □

4 Intersection of Two r.e. Sets

Proposition 1. *If A and B are r.e., then $A \cap B$ is r.e.*

Proof. Run enumerators for A and B in parallel. Whenever an element appears in both enumerations, output it. □

Theorem 3. *There is no algorithm that decides, given indices e, f, whether*

$$|W_e \cap W_f| = \infty.$$

Proof. Fix an infinite r.e. set A and an index a with $W_a = A$. For any e,

$$|W_e \cap A| = \infty \iff |W_e \cap W_a| = \infty.$$

By the previous theorem, the left-hand property is undecidable, hence so is the right-hand one. □

5 Unifying Perspective

All the undecidability results above reduce to the same fundamental fact:

> *There is no algorithm to decide whether a recursively enumerable set is infinite.*

Indeed:

- "Does W_e contain infinitely many primes?"
- "Does W_e contain infinitely many elements of a fixed infinite r.e. set?"
- "Is $W_e \cap W_f$ infinite?"

are all instances of asking whether a certain r.e. set is infinite.

This reflects the fact that infiniteness is a Π^0_2 property,

$$|W| = \infty \iff \forall n \, \exists x > n \ (x \in W),$$

which cannot be decided from finite initial segments of an enumeration.

6 Conclusion

The closure of r.e. sets under intersection does not contradict the undecidability of infiniteness. While membership can be semi-decided, global properties depending on infinite behavior cannot. All the problems considered here ultimately encode the halting problem.