TWO INEQUALITIES
AND A FUNCTIONAL EQUATION

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Introduction

This is a list of three related problems involving the expressions $E_1 = f(x+y)f(x-y)$ and $E_2 = f(x)^2 - f(y)^2$. Each of them explores the consequences of three different hypothesis, namely, $E_1 \leq E_2$, $E_1 = E_2$, and $E_1 \geq E_2$ respectively. The first one I found in Math StachExchange together with a solution already posted, I just rewrote the results with some editing for clarity. The second one is in fact a functional equation, and the third one I found posted in Quora without any solution posted after several months. The solutions posted here were obtained with AI assistance in the way I will make more clear for each individual problem. The models used were ChatGPT o3, and ChatGPT-5.

1. First Inequality

Problem [KMO 1987]. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, such that for all $x, y \in \mathbb{R}$,

$$f(x+y)f(x-y) \leq f(x)^2 - f(y)^2.$$

Prove:

$$f(x) = -f(-x)$$

and

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2.$$

Solution. Letting $x = 0$ we get $f(y)f(-y) \leq f(0)^2 \leq 0$, hence $f(y)$ and $f(-y)$ are both zero, or they have opposite signs. From here $f(x) = -f(-x)$ (i.e., f is odd) follows.

For the second equality, the oddness of f implies $f(x-y) = -f(y-x)$, hence

$$\begin{aligned} f(x+y)f(x-y) &= -f(y+x)f(y-x) \\ &\geq -(f(y)^2 - f(x)^2) \\ &= f(x)^2 - f(y)^2, \end{aligned}$$

which together with the hypothesis $f(x+y)f(x-y) \leq f(x)^2 - f(y)^2$ implies the equality.[1]

□

Remarks. In this problem I didn't use AI assistance, I just rewrote the known solution that appeared in Math StackExchange (slightly edited for clarity).

[1]Clement Yung (https://math.stackexchange.com/users/620517/clement-yung), Functional inequality $f(x+y)f(x-y) \leq f^2(x) - f^2(y)$, URL (version: 2020-05-15): https://math.stackexchange.com/q/3675646

2. Functional Equation

Problem: Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2.$$

Step 1. Basic consequences.

Set $y = 0$:

$$f(x)^2 = f(x)^2 - f(0)^2 \;\Rightarrow\; f(0) = 0.$$

Set $x = 0$:

$$f(y)f(-y) = -f(y)^2 \;\Rightarrow\; f(-y) = -f(y) \quad \text{for all } y,$$

so f is odd. Denote the equation by:

$$(P(x,y)) \qquad f(x+y)f(x-y) = f(x)^2 - f(y)^2.$$

Step 2. Cos–sin decomposition.

Fix y and choose x with $f(x) \neq 0$. Define:

$$C(y) := \frac{f(x+y) + f(x-y)}{2f(x)}, \quad S(x) := \frac{f(x+y) - f(x-y)}{2f(y)} \quad (y \neq 0).$$

A standard manipulation using (P) shows that:

$$f(x+y) = f(x)C(y) + f(y)S(x)$$

for all x, y. Plugging into $P(x,y)$ and comparing the coefficients of $f(x)^2$ and $f(y)^2$ yields that there exists a constant $\lambda \in \mathbb{R}$ such that:

$$C(t)^2 - \lambda\, S(t)^2 = 1,$$

and C, S satisfy:

$$C(s+t) = C(s)C(t) + \lambda\, S(s)S(t),$$
$$S(s+t) = S(s)C(t) + C(s)S(t),$$

with $C(0) = 1, S(0) = 0, C$ even, S odd.

Step 3. Classification of $(\mathbf{C}, \mathbf{S})$.

From the above, C, S form a one-parameter subgroup and are determined by an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(x+y) = \alpha(x) + \alpha(y)$ for all $x, y \in \mathbb{R}$.

The possible cases are:

- Elliptic case ($\lambda = 1$):
$$C(t) = \cos(\alpha(t)), \quad S(t) = \sin(\alpha(t)).$$
- Hyperbolic case ($\lambda = -1$):
$$C(t) = \cosh(\alpha(t)), \quad S(t) = \sinh(\alpha(t)).$$

- Parabolic case ($\lambda = 0$):

$$C(t) = 1, \quad S(t) = \alpha(t).$$

Step 4. General solutions.

From $f(x+y) = f(x)C(y) + f(y)S(x)$ and the oddness of f, the general solutions are:

$$\begin{aligned} &\boxed{f(x) = 0} && \text{(the zero function)},\\ &\boxed{f(x) = A\sin(\alpha(x))} && \text{(elliptic case)},\\ &\boxed{f(x) = A\sinh(\alpha(x))} && \text{(hyperbolic case)},\\ &\boxed{f(x) = A\,\alpha(x)} && \text{(parabolic case)}, \end{aligned}$$

Where ($A \in \mathbb{R} \setminus \{0\}$, and α is additive).

It is immediate from:

$$\begin{aligned} &\sin(u+v)\sin(u-v) = \sin^2 u - \sin^2 v,\\ &\sinh(u+v)\sinh(u-v) = \sinh^2 u - \sinh^2 v,\\ &(u+v)(u-v) = u^2 - v^2, \end{aligned}$$

that these satisfy the equation.

Remark. If we assume any mild regularity (measurability, local boundedness, or continuity at one point), then $\alpha(x) = kx$ for some constant k, and the solutions are:

$$\boxed{f(x) \equiv 0, \quad f(x) = A\sin(kx), \quad f(x) = A\sinh(kx), \quad f(x) = Akx}.$$

Remarks. Here the AI assistant produced solutions that look correct, although a question remains about whether those are all possible solutions to the functional equation. I noticed that in one of the iterations the AI omitted the additive function α and claimed that the general solution is the one presented after assuming "mild regularity." I also noted that it is not clear to me that local boundedness would be enough to rule out $f(x) = A\sin(\alpha(x))$ with non-linear α as a solution since the *sin* function is always bonded regardless of its argument. In general this section may require more scrutiny to make sure that the AI assistant didn't make any fatal mistake, for now I leave it as it was generated, with some editing for clarity.

3. Second Inequality

Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, such that for all $x, y \in \mathbb{R}$,

$$f(x+y)f(x-y) \geq f(x)^2 - f(y)^2.$$

Assume that the inequality is strict for some $x_0, y_0 \in \mathbb{R}$. Show that either $f(x) \geq 0$ for every $x \in \mathbb{R}$ or $f(x) \leq 0$ for every $x \in \mathbb{R}$.[2]

Solution. We split into two cases.

Case 1: $f(0) \neq 0$**.** Setting $x = y = t/2$ gives $f(t)f(0) \geq 0$ for all t, hence f has a constant sign (the sign of $f(0)$).

Case 2: $f(0) = 0$**.** First, with $(x, y) = (t, -t)$ we obtain

$$0 = f(0)f(2t) \;\geq\; f(t)^2 - f(-t)^2,$$

and swapping $t \mapsto -t$ gives the reverse inequality, hence

$$|f(t)| = |f(-t)| \qquad (\forall t \in \mathbb{R}). \tag{1}$$

Claim 1. If f is odd, i.e. $f(-t) = -f(t)$ for all t, then the displayed inequality holds with equality for all x, y. Indeed, swapping x and y gives

$$f(y+x)f(y-x) \;\geq\; f(y)^2 - f(x)^2.$$

Since $y + x = x + y$ and oddness gives $f(y-x) = -f(x-y)$, we get

$$-f(x+y)f(x-y) \;\geq\; f(y)^2 - f(x)^2,$$

equivalently $f(x+y)f(x-y) \leq f(x)^2 - f(y)^2$, which combined with the original inequality yields equality everywhere. In particular, odd f cannot satisfy the *strict somewhere* hypothesis.

Consequently, by strictness there exists some a with $f(-a) \neq -f(a)$, and by (1) this forces

$$f(-a) = f(a) \neq 0. \tag{2}$$

Next, assume for contradiction that f changes sign. Choose b with $f(b)$ of opposite sign to $f(a)$, so $f(a)f(b) < 0$. Define

$$u = \frac{a+b}{2}, \qquad v = \frac{a-b}{2} \qquad (\text{so } u + v = a,\ u - v = b).$$

Apply the hypothesis at $(x, y) = (u, v)$:

$$f(a)f(b) \;\geq\; f(u)^2 - f(v)^2. \tag{3}$$

Since $f(a)f(b) < 0$, (3) implies

$$f(u)^2 - f(v)^2 \;<\; 0. \tag{4}$$

[2]Proposed by Amit Kumar in Quora - https://qr.ae/pYbVKX

Next apply the hypothesis at $(x, y) = (-v,\ u)$, so that $(-v)+u = b$ and $(-v)-u = -a$:

$$f(b)f(-a) \ \geq\ f(-v)^2 - f(u)^2.$$

Using (2) and (1) this becomes

$$f(a)f(b) \ \geq\ f(v)^2 - f(u)^2. \tag{5}$$

Again $f(a)f(b) < 0$, so (5) yields

$$f(v)^2 - f(u)^2 \ <\ 0. \tag{6}$$

The strict inequalities (4) and (6) are incompatible, a contradiction. Hence f cannot change sign.

Combining both cases, f has a constant sign on $\mathbb{R}$. Finally, the strictness assumption rules out the identically zero function (which would give equality everywhere), so one of $f \geq 0$ or $f \leq 0$ holds. □

Remark. The role of the *strict somewhere* hypothesis is exactly to exclude the globally odd "borderline" solutions (e.g. $f(x) = x$ or $f(x) = \sin x$), for which the inequality holds as an equality for all pairs (x, y) and sign changes do occur. In Case 2, strictness guarantees the existence of a nonzero point a with $f(-a) = f(a)$, which is the only place where parity information is used in the contradiction.

Remarks. This version of the problem was the one that required more iterations until getting the proposed solution presented here. The first solution produced by the AI was clearly wrong, since it made a serious mistake handling multiple inequalities, basically claiming that the combination of two inequalities $P > Q$ and $P > R$ implied $Q > R$, which is clearly wrong. When I see obvious errors like this I inquire the AI to give more details of how it made such inference, because that may reveal a subjacent reasoning that may still be salvageable, but not in this case, the AI just had made an unfixable mistake, apologized, and proceeded to produce another "solution." The following outputs still had issues that required further fixing and new iteractions. From some session to the next the AI seemed to forget intermediate results that were essential to obtain a final solution, such as the proof that in Case 2 the function cannot be odd because that would imply a violation of the "strict somewhere" hypothesis. At a point the AI was already getting close to giving up claiming something of the sort "I know this is not a full solution, but it is all I can do," until I pointed out that the intermediate result I just mentioned (Claim 1) was the missing piece in the reasoning. This allowed the AI to produce the solution presented here, and caused it to stress why and in which way the strict somewhere hypothesis was essential to obtain a solution, as indicated in the final remark.