SPLIT SQUARES

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Introduction

Here we examine a problem, its original human-provided solution, an AI-provided solution, and a new version of the original solution after being revised by the AI assistant. The AI model used was ChatGTP-5.

Problem and Original Solution

Problem. Prove that there are infinitely many squares not multiple of 10 whose representation in base 10 can be split into two squares. For instance $7^2 = 49$ can be split $4|9$, where 4 and 9 are squares ($4 = 2^2$, $9 = 3^2$); $13^2 = 169$ can be split $16|9$, again two squares, etc. (we exclude multiples of 10 in order to avoid trivial answers like the infinite sequence $49 = 4|9$, $4900 = 4|900$, $490000 = 4|90000$, etc.).

Original Solution. The fact that the decimal representation of a square z^2 (not a multiple of 10) is the concatenation of two squares x^2 and y^2 can be expressed with the following system of equation and inequality:

$$\begin{aligned} &10^n x^2 + y^2 = z^2 \\ &10^{n-1} < y^2 < 10^n \,, \end{aligned} \tag{1}$$

where x, y, z, n must be positive integers and y and z are not multiple of 10. So we need to prove that (1) has infinitely many solutions. In fact we will prove more, namely that for any given positive integer x, (1) has infinitely many solutions. So in the following we assume that x is any fix given positive integer.

We start by rewriting the equation in the following way:

$$10^n x^2 = z^2 - y^2 = (z + y)(z - y).$$

Since the left hand side is even, y and z must have the same parity, so the two factors on the right must be even and we can write $z + y = 2p$, $z - y = 2q$ for some positive integers p and q. Then we have $z = p + q$, $y = p - q$, and $10^n x^2 = 4pq$, so $q = 10^n x^2/(4p)$. Hence the inequality can be written like this:

$$10^{(n-1)/2} < p - \frac{10^n x^2}{4p} < 10^{n/2} \,.$$

The expression $f(p) = p - 10^n x^2/(4p)$ is an increasing function of p, and verifies $f(10^{n/2} b_1/2) = 10^{(n-1)/2}$ and $f(10^{n/2} b_2/2) = 10^{n/2}$, where

$$b_1 = 1/\sqrt{10} + \sqrt{1/10 + x^2} \quad \text{ and } \quad b_2 = 1 + \sqrt{1 + x^2}\,.$$

So the inequality becomes

$$\frac{10^{n/2}}{2} b_1 < p < \frac{10^{n/2}}{2} b_2\,.$$

Taking decimal logarithms we get

$$\frac{n}{2} + \log_{10} b_1 - \log_{10} 2 < \log_{10} p < \frac{n}{2} + \log_{10} b_2 - \log_{10} 2$$

or equivalently

$$n < 2 \log_{10} p + \alpha < n + \beta\,,$$

where, $\alpha = 2 \log_{10}(2/b_1)$, $\beta = 2 \log_{10}(b_2/b_1)$. We note that α and β depend only on x, but not on p or n, and also that $\beta > 0$. Also recall that $4p$ must be a divisor of $10^n x^2$, and $p \pm q$ should not be a multiple of 10. These conditions are met if we set $n > 2$ and $p = 5^k$ for some $0 \leq k < n$. Then the inequality becomes

$$n < 2k \log_{10} 5 + \alpha < n + \beta\,,$$

or equivalently

$$\begin{aligned} n &= \lfloor 2k \log_{10} 5 + \alpha \rfloor\,, \\ 0 &< \{\!\{ 2k \log_{10} 5 + \alpha \}\!\} < \beta\,, \end{aligned}$$

where $\lfloor t \rfloor$ = integer part of t, $\{\!\{ t \}\!\} = t - \lfloor t \rfloor$ = fractional part of t. Since $2 \log_{10} 5 > 1$, the condition $k < n$ will be satisfied for every k large enough. On the other hand since the integer multiples of an irrational number are dense modulo 1, and $2 \log_{10} 5$ is indeed irrational, we have that the fractional part of $2k \log_{10} 5$ is in $(0, \beta)$ for infinitely many values of k. So since all the conditions are satisfied for infinitely many values of k, we have that (1) has infinitely many solutions.

The argument used here can be used to search numerically for specific solutions of (1). The idea is to pick any positive integer x and assign values $1, 2, 3, \ldots$ to k checking whether the following conditions are verified:

$$\begin{aligned} n &= \lfloor 2k \log_{10} 5 + \alpha \rfloor > k\,, \\ 0 &< \{\!\{ 2k \log_{10} 5 + \alpha \}\!\} < \beta\,, \end{aligned}$$

Example: First we pick any positive value for x, say $x = 1$. Next we compute $2 \log_{10}(5) = 1.397940008\ldots$, $\alpha = 0.3317713906\ldots$, $\beta = 0.4952627696\ldots$. Finally we search for values of k such that

$$\begin{aligned} n &= \lfloor 1.397940008k + 0.3317713906 \rfloor > k\,, \\ 0 &< \{\!\{ 1.397940008k + 0.3317713906 \}\!\} < 0.4952627696\,. \end{aligned}$$

For instance, for $k = 2$ we have $1.397940008k + 0.3317713906 = 3.127651407$, so $k = 2$ satisfies the conditions, yielding the solution $n = 3$, $p = 5^2 = 25$, $q = 10^3/(4 \cdot 25) = 10$, $y = 25 - 10 = 15$, $z = 25 + 10 = 35$. So $y^2 = 225$, $z^2 = 1225$. Hence $35^2 = 1225 = 1|225$ can be split into $1 = 1^2$ and $225 = 15^2$.

Solution Given by the AI Assistant

Theorem 1. *There are infinitely many perfect squares (not divisible by* 10*) whose base–*10 *expansion can be split into two consecutive blocks, each of which is a perfect square.*

Proof. We produce infinitely many examples with a one-digit split at the end. Let $c \in \{1, 4, 9\}$ be a one-digit square. We seek integers x, y such that

$$x^2 = 10y^2 + c. \tag{1}$$

If (1) holds, then dividing by 10 shows that x^2 has quotient y^2 and remainder c, hence its decimal expansion is exactly the concatenation

$$x^2 \;=\; \overline{\text{digits of } y^2} \;\mid\; \overline{c}.$$

Moreover, because $x^2 \equiv c \pmod{10}$ with $c \in \{1, 4, 9\}$, none of these squares is divisible by 10.

Equation (1) is a Pell-type equation

$$x^2 - 10y^2 = c.$$

Since $19^2 - 10 \cdot 6^2 = 1$, the unit

$$\alpha := 19 + 6\sqrt{10} \in \mathbb{Z}[\sqrt{10}]$$

has norm $N(\alpha) = 1$. If (x_0, y_0) is any integer solution to $x^2 - 10y^2 = c$, then for every $t \geq 0$ the pair (x_t, y_t) defined by

$$x_t + y_t\sqrt{10} = (x_0 + y_0\sqrt{10})\,\alpha^t$$

also satisfies $x_t^2 - 10y_t^2 = c$ (norms multiply). Thus each initial solution produces an infinite family.

We now give one seed for each $c \in \{1, 4, 9\}$:

$$\begin{aligned} c = 1: &\quad (x_0, y_0) = (19, 6), \quad 19^2 - 10 \cdot 6^2 = 1; \\ c = 4: &\quad (x_0, y_0) = (2, 0), \quad 2^2 - 10 \cdot 0^2 = 4; \\ c = 9: &\quad (x_0, y_0) = (7, 2), \quad 7^2 - 10 \cdot 2^2 = 9. \end{aligned}$$

Therefore each choice of c yields infinitely many solutions to (1), hence infinitely many squares with a split into two squares.

For explicit recurrences, note that

$$(x + y\sqrt{10})\alpha = (19x + 60y) + (6x + 19y)\sqrt{10},$$

so from any solution (x, y) to $x^2 - 10y^2 = c$ we obtain a new one (x', y') via

$$x' = 19x + 60y, \qquad y' = 6x + 19y. \tag{2}$$

Iterating (2) gives an infinite sequence of examples.

Examples.

- $c = 9$: $(x, y) = (7, 2) \mapsto (253, 80) \mapsto (9607, 3038) \mapsto \cdots$

  $7^2 = 49 = 4 \mid 9, \quad 253^2 = 64009 = 6400 \mid 9, \quad 9607^2 = 92{,}294{,}449 = 9{,}229{,}444 \mid 9.$
- $c = 4$: $(x, y) = (2, 0) \mapsto (38, 12) \mapsto (1442, 456) \mapsto \cdots$

  $$38^2 = 1444 = 144 \mid 4, \qquad 1442^2 = 2{,}079{,}364 = 207{,}936 \mid 4.$$
- $c = 1$: $(x, y) = (19, 6) \mapsto (721, 228) \mapsto \cdots$

  $$19^2 = 361 = 36 \mid 1, \qquad 721^2 = 519{,}841 = 51{,}984 \mid 1.$$

Each listed square ends in $1, 4$, or 9, hence none is a multiple of 10. This completes the proof. $\square$

Remarks. We notice the following:

(1) The AI proof uses an algebraic approach, while the original used a density argument based in the Kronecker–Weyl equidistribution theorem.

(2) While the AI solution solves the problem as stated, the original solution provides a stronger result, namely that there are infinitely many examples with any prescribed left block x^2, which is strictly stronger than the Pell-based "last-digit block" constructions. The AI assistant pointed out after revision that that the strengthening is worth calling out in the statement.

(3) The AI assistant made a few suggestions (reflected in the revised solution below).
   (a) State the "strong" result up front.
   (b) Add the one-line derivative for $f'(p)$.
   (c) Spell out the "not multiple of 10" check once.
   (d) A few tiny wording/LaTeX nits.
   (e) Optional: you don't actually need logs; equivalently, you can write $n = \lfloor 2 \log_{10} p + \alpha \rfloor$ and demand $0 < \{\!\{2k \log_{10} 5 + \alpha\}\!\} < \beta$ then substitute $p = 5^k$. The log step is fine as is, though.

Suppose the decimal expansion of a square z^2 can be written as the concatenation of two smaller squares x^2 and y^2. This is equivalent to the system

$$\begin{aligned} z^2 &= 10^n x^2 + y^2, \\ 10^{n-1} &< y^2 < 10^n, \end{aligned} \tag{1}$$

where x, y, z, n are positive integers and y, z are not divisible by 10. We shall prove that (1) has infinitely many solutions. In fact, for any fixed $x \geq 1$ there are infinitely many such solutions.

Step 1. Factorization. Rewriting the first equation,

$$10^n x^2 = z^2 - y^2 = (z+y)(z-y).$$

Since $10^n x^2$ is even, z and y have the same parity. Thus we may set

$$z + y = 2p, \qquad z - y = 2q,$$

with $p, q \in \mathbb{Z}_{>0}$. Then $z = p + q$, $y = p - q$, and

$$4pq = 10^n x^2.$$

Hence

$$q = \frac{10^n x^2}{4p}, \qquad y = p - q = f(p).$$

Step 2. Inequality. The condition on y^2 is equivalent to

$$10^{(n-1)/2} < f(p) < 10^{n/2}, \qquad f(p) = p - \frac{10^n x^2}{4p}.$$

Note that f is strictly increasing since

$$f'(p) = 1 + \frac{10^n x^2}{4p^2} > 0.$$

Step 3. Scaling. Let $p = \frac{10^{n/2}}{2} b$. Then

$$f(p) = 10^{n/2} \cdot \frac{b^2 - x^2}{2b}.$$

The inequalities $f(p) = 10^{(n-1)/2}$ and $f(p) = 10^{n/2}$ yield two constants

$$b_1 = \tfrac{1}{\sqrt{10}} + \sqrt{\tfrac{1}{10} + x^2}, \qquad b_2 = 1 + \sqrt{1 + x^2},$$

such that the condition on p is equivalent to

$$\frac{10^{n/2}}{2} b_1 < p < \frac{10^{n/2}}{2} b_2.$$

Step 4. Logarithmic formulation. Taking base–10 logarithms, we obtain

$$n < 2 \log_{10} p + \alpha < n + \beta,$$

where

$$\alpha = 2\log_{10}\left(\tfrac{2}{b_1}\right), \qquad \beta = 2\log_{10}\left(\tfrac{b_2}{b_1}\right) > 0.$$

Step 5. Choice of p. Choose $p = 5^k$ with $0 \le k < n$. Then

$$n = \lfloor 2k\log_{10} 5 + \alpha \rfloor, \qquad 0 < \{2k\log_{10} 5 + \alpha\} < \beta,$$

where $\{t\}$ denotes the fractional part. Since $2\log_{10} 5 > 1$, the condition $k < n$ holds for all large k. Because $2\log_{10} 5$ is irrational, the sequence $\{2k\log_{10} 5\}$ is dense modulo 1. Thus the inequality above is satisfied for infinitely many k.

Step 6. Not multiples of 10. From $4pq = 10^n x^2$ with $p = 5^k$ we have

$$q = 2^{n-2} 5^{n-k} x^2.$$

So q is even, p is odd, hence $y = p - q$ and $z = p + q$ are odd. Thus neither y nor z is divisible by 2. If $k \ge 1$, then p, q are multiples of 5, hence $y, z \equiv 0 \pmod 5$ but not $\pmod{10}$. If $k = 0$, then $p \equiv \pm 1 \pmod 5$ while $q \equiv 0 \pmod 5$, so again y, z are not multiples of 5. Therefore in all cases y, z are not divisible by 10.

Step 7. Example. Take $x = 1$. Then

$$2\log_{10} 5 \approx 1.39794, \quad \alpha \approx 0.33177, \quad \beta \approx 0.49526.$$

For $k = 2$ we obtain $n = 3$, $p = 25$, $q = 10$, so $y = 15$, $z = 35$. Hence

$$35^2 = 1225 = 1 \mid 225,$$

which is the concatenation of 1^2 and 15^2.

Conclusion. Thus for each fixed x there exist infinitely many solutions (n, y, z) to (1), giving infinitely many squares not divisible by 10 whose decimal expansion splits into two squares.