Nonrelativistic and Relativistic Quantum Mechanics: Incompatibilities, Axiomatic Frameworks, CPT, and Spin–Statistics

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Abstract

This note summarizes the ways ordinary nonrelativistic quantum mechanics differs from special relativity; describes axiomatic approaches to nonrelativistic quantum mechanics, relativistic single-particle theory, and relativistic quantum field theory; and explains how CPT and spin–statistics become theorems in suitable axiomatic quantum field theories. The emphasis is on separating mathematical assumptions, rigorous conclusions, and physical interpretations.

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1 Nonrelativistic quantum mechanics versus special relativity

Nonrelativistic quantum mechanics (NRQM) is an effective theory, accurate in an appropriate low-velocity and low-energy regime. It is not Lorentz covariant as an exact theory. Several related distinctions explain why.

1.1 Galilean versus Poincaré symmetry

The free Schrödinger equation is

$$i\hbar\partial_t\psi(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t). \tag{1}$$

It is covariant under Galilean boosts

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t,$$

provided the wavefunction is assigned the appropriate mass-dependent phase. It is not covariant under Lorentz boosts, which mix time and space:

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right), \qquad \gamma = (1 - v^2/c^2)^{-1/2}.$$

Accordingly, ordinary NRQM uses a global time parameter, while relativity does not privilege a universal simultaneity relation. In relativistic theories, quantum states can instead be associated with spacelike hypersurfaces, or one can formulate the theory directly in terms of local spacetime observables.

1.2 The energy–momentum relation

The free NRQM Hamiltonian is

$$E_{\mathrm{NR}} = \frac{\mathbf{p}^2}{2m}.$$

Relativistic kinematics requires

$$E^2 = c^2\mathbf{p}^2 + m^2c^4, \qquad E = mc^2 + \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3c^2} + O(|\mathbf{p}|^6).$$

After removing the constant rest energy, Schrödinger dynamics is the leading low-momentum approximation. Simply replacing the Hamiltonian by

$$H = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}$$

gives the correct free-particle dispersion relation but does not, by itself, solve the problems of relativistic locality and interacting particle creation. As an operator in position space, the square-root Hamiltonian is nonlocal.

1.3 Instantaneous spreading and causality

The one-dimensional free Schrödinger propagator, for $t > 0$, is

$$K(x,t;x',0) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im(x-x')^2}{2\hbar t}\right). \tag{2}$$

It is nonzero for every finite $x - x'$ and $t > 0$. A generic initially compactly supported wavefunction therefore has immediately nonzero tails arbitrarily far away; there is no sharp propagation front bounded by c. This exhibits the absence of a relativistic causal structure in ordinary Schrödinger evolution.

Important qualification. Nonzero wavefunction amplitude at a distant point is not by itself proof that a controllable faster-than-light signal can be sent. Relativistic locality is more precisely expressed by commutation of local observables at spacelike separation, as described below.

1.4 Fixed particle number and particle production

A standard single- or fixed-N-particle Schrödinger theory does not intrinsically include processes in which particles are created or destroyed. Relativistic reactions such as

$$\gamma + \gamma \longrightarrow e^- + e^+$$

(with suitable kinematics) motivate a description with sectors of variable particle number, often organized into Fock space

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots .$$

Fixed-particle-number relativistic theories can be constructed, particularly for free particles, but local relativistic interactions naturally lead to quantum field theory (QFT).

1.5 Relativistic wave equations and measurement

Two historically important equations are the Klein–Gordon equation,

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi = 0,$$

and the Dirac equation,

$$(i\hbar c\gamma^\mu \partial_\mu - mc^2)\psi = 0.$$

The former does not possess a generally positive-definite standard conserved single-particle density; the latter admits a positive density in its first-quantized interpretation, but negative-frequency solutions and interactions point toward a field-theoretic framework.

Textbook instantaneous collapse poses an additional interpretive tension with relativity of simultaneity. It does *not* entail that entanglement permits superluminal messaging: the operational predictions satisfy no-signalling constraints.

2 Axiomatizing nonrelativistic quantum mechanics

A standard Hilbert-space scheme may be stated as follows. These are structural postulates, rather than a uniquely canonical minimal axiom list.

N1. States. A system is assigned a complex Hilbert space $\mathcal{H}$. Pure states are rays (or rank-one projections), and mixed states are positive trace-class operators ρ with $\operatorname{Tr}\rho = 1$.

N2. Observables. Sharp observables are self-adjoint operators A, with projection-valued spectral measures P_A.

N3. Probabilities. The Born rule is

$$\Pr_\rho(A \in \Delta) = \operatorname{Tr}(\rho P_A(\Delta)).$$

More general measurements may be represented by positive operator-valued measures.

N4. Dynamics. For a closed system, time evolution is a strongly continuous one-parameter unitary group,

$$U(t) = e^{-itH/\hbar},$$

with self-adjoint generator H (Stone's theorem).

N5. Composition. Distinguishable subsystems combine by $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Identical bosons and fermions use the appropriate (anti)symmetric sectors; the usual NRQM axioms alone do not derive which spin must use which sector.

N6. Nonrelativistic spacetime symmetry. The Galilei group acts through a projective unitary representation (or an ordinary representation of its central extension). For a standard massive free spinless particle,

$$[X_i, P_j] = i\hbar\delta_{ij}\mathbf{1}, \qquad H = \frac{\mathbf{P}^2}{2m} + E_0.$$

The first several postulates are quantum-mechanical rather than specifically nonrelativistic. The spacetime representation and physical dynamics supply the relevant distinction.

3 Axiomatizing relativistic single-particle quantum mechanics

For a free elementary relativistic particle, one retains the Hilbert-space and probability structure but replaces Galilei symmetry by the Poincaré group. A possible scheme is:

R1. States are rays in a positive-definite Hilbert space and probabilities obey the Born rule.

R2. A strongly continuous (projective) unitary representation of the proper orthochronous Poincaré group, or the corresponding covering group, implements spacetime symmetry.

R3. The joint energy–momentum spectrum lies in the closed forward light cone $\overline{V}_+$ (the *spectrum condition*).

R4. An elementary particle is associated with an irreducible positive-energy representation, classified (in the massive case) by mass and spin.

For a massive particle,

$$P_\mu P^\mu = m^2c^2, \qquad E^2 = c^2\mathbf{p}^2 + m^2c^4.$$

This is the setting of Wigner's classification. It provides a mathematically sound relativistic theory of free particles, but Poincaré covariance and positive energy alone do not deliver a complete theory of local particle interactions or the spin–statistics connection.

4 Axiomatic relativistic quantum field theory

4.1 The Wightman framework

Wightman-type axioms specify quantum fields as operator-valued tempered distributions on a positive-definite Hilbert space with a vacuum state. In a standard formulation the essential ingredients include:

1. a common invariant dense domain and distributional regularity of the fields;
2. a strongly continuous unitary representation of the proper orthochronous Poincaré group and covariant transformation of fields;
3. the spectrum condition, $\mathrm{spec}(P^\mu) \subseteq \overline{V}_+$;
4. a Poincaré-invariant vacuum vector Ω, with an appropriate vacuum cyclicity assumption;
5. local (graded) commutativity of fields at spacelike separation, with the proper field-type conventions.

The axioms constrain entire families of theories; they do not specify one particular Lagrangian.

4.2 The Haag–Kastler framework

Algebraic QFT assigns each suitable spacetime region O an algebra of local observables, $O \mapsto \mathcal{A}(O)$. Core structural properties include:

Isotony $O_1 \subseteq O_2 \Longrightarrow \mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$.

Locality If O_1 and O_2 are spacelike separated, then

$$[A, B] = 0 \quad (A \in \mathcal{A}(O_1),\ B \in \mathcal{A}(O_2)).$$

Covariance $U(g)\mathcal{A}(O)U(g)^{-1} = \mathcal{A}(gO)$.

Positive energy
The translation generators satisfy the spectrum condition.

Vacuum structure
An appropriate vacuum representation includes a Poincaré-invariant vector; additional uniqueness or cyclicity conditions may be imposed.

This observable-centered formulation does not initially require choosing charged field operators. To prove strong versions of CPT or spin–statistics one generally needs additional hypotheses beyond a minimal list of net axioms.

5 The CPT theorem

5.1 Statement and assumptions

Theorem 5.1 (CPT, schematic Wightman form)**.** *For a local relativistic QFT in four-dimensional Minkowski spacetime satisfying the standard Wightman hypotheses (including Poincaré covariance, positive-energy spectrum, vacuum structure, positivity, regularity, and suitable local commutativity), there exists an antiunitary CPT symmetry Θ with the expected action on local fields and states.*

For a Hermitian scalar field, suitable conventions give

$$\Theta\phi(x)\Theta^{-1} = \phi(-x).$$

For charged or spinorial fields the right-hand side involves adjoints or charge-conjugate fields and spin-dependent matrices. With the standard antiunitary convention, the relation to proper orthochronous Poincaré transformations can be written

$$\Theta U(a,\Lambda)\Theta^{-1} = U(-a,\Lambda).$$

CPT invariance does *not* assert separate invariance under C, P, or T.

5.2 Why the proof is possible

Let

$$W_n(x_1,\dots,x_n) = \langle\Omega, \phi(x_1)\cdots\phi(x_n)\Omega\rangle$$

be vacuum n-point distributions. The proof has the following architecture:

1. The positive-energy spectrum condition makes Wightman distributions boundary values of analytic functions in suitable complex spacetime tubes.

2. Covariance permits analytic continuation to the complex Lorentz group. Full spacetime inversion is accessible through complex Lorentz transformations, although not through the connected real proper orthochronous group.

3. Locality relates reversed field orderings at spacelike-separated configurations (and then, by analytic continuation, the relevant correlation functions more generally).

4. These identities yield the CPT relation between correlation functions. Reconstruction or a density argument implements it by an antiunitary operator on the Hilbert space.

Jost's formulation identifies a weaker condition, *weak local commutativity at Jost points*, that is central to a precise CPT criterion. Thus the exact list of sufficient hypotheses can be sharpened beyond this summary.

6 The spin–statistics theorem

Theorem 6.1 (Spin–statistics, schematic four-dimensional form)**.** *In an appropriate positive-metric, positive-energy, local, Poincaré-covariant quantum field theory in* $3+1$ *dimensions, nontrivial integer-spin fields have Bose-type spacelike exchange relations and nontrivial half-integer-spin fields have Fermi-type spacelike exchange relations, with the usual qualifications for charged and multicomponent fields.*

In a simplified same-field notation, the exchange sign is

$$\phi_a(x)\phi_b(y) = \eta\,\phi_b(y)\phi_a(x) \quad \text{for spacelike-separated } x, y, \qquad \eta = (-1)^{2s}.$$

The precise theorem is formulated for the correct covariant field components and graded commutation relations. It is not a universal scalar formula for every arbitrary observable or charged field.

6.1 Logical ingredients

The crucial combination is

Lorentz covariance + spectrum condition + locality + Hilbert-space positivity.

Analytic continuation and exchange of spacelike-separated field insertions impose relations on correlation functions. Positivity then excludes nontrivial fields carrying the wrong spin–statistics assignment. This explains why the theorem cannot be derived from the bare NRQM Hilbert-space axioms or from relativistic single-particle representation theory alone.

6.2 Scope and exceptions

- In NRQM the assignment of bosonic or fermionic symmetry to different spin species is an additional physical stipulation, not a consequence of Galilean invariance alone.
- In gauge-fixed descriptions, unphysical ghost fields may have unconventional spin–statistics assignments because the auxiliary state space or notion of physical field does not satisfy the standard positivity hypotheses.
- In two spatial dimensions, more general braid-group statistics (anyons) are possible, so the usual three-spatial-dimensional Bose/Fermi dichotomy does not carry over unchanged.

7 The algebraic approach to statistics and CPT

In Haag–Kastler theory, observables in spacelike-separated regions commute. Fermionic charged fields need not themselves belong to the observable net. Instead, superselection sectors of representations can encode charge and exchange statistics.

The Doplicher–Haag–Roberts (DHR) analysis, under suitable localization and other hypotheses, associates statistics data to sectors and explains the Bose/Fermi alternative in four spacetime dimensions. Further assumptions connect this intrinsic statistical grading to the representation of 2π rotations.

A notable example is the algebraic spin–statistics and PCT work of Guido and Longo. Their results use hypotheses related to the geometric action of modular groups of wedge algebras (closely connected with the Bisognano–Wichmann property). One should therefore not say that the minimal net axioms alone yield all versions of these theorems without further conditions.

8 Formal logic and the consistency question

The word *axiomatic* has at least two useful senses:

1. a mathematically rigorous family of postulates, formulated with Hilbert spaces, operators, Lie groups, and operator algebras;
2. a fully formal deductive theory inside a foundation such as ZFC, with definitions and proofs encoded in a precise logical language.

The principal axiomatic QFT results are rigorous theorems in the first sense and can, in principle, be represented in the second. If W denotes a suitable formalized set of Wightman axioms, one may write schematically

$$W \vdash \mathrm{CPT}, \qquad W \vdash \mathrm{SpinStatistics}.$$

These implications must be read with their exact assumptions in mind. The question of whether a particular physically realistic interacting four-dimensional QFT satisfies all relevant axioms is distinct from a conditional theorem asserting what follows *if* the axioms hold.

8.1 An elementary incompatibility illustration

For a standard massive free nonrelativistic particle, suppose

$$H = \frac{\mathbf{P}^2}{2m} + E_0.$$

If the *same* observables also satisfy the relativistic mass shell relation,

$$H^2 = c^2\mathbf{P}^2 + m^2c^4,$$

then

$$\left(\frac{\mathbf{P}^2}{2m} + E_0\right)^2 = c^2\mathbf{P}^2 + m^2c^4.$$

This cannot hold on a continuum of arbitrary momentum values: the left side has a nonzero quadratic term in $\mathbf{P}^2$ whereas the right side is only linear in $\mathbf{P}^2$. The contradiction concerns *simultaneously imposing both physical dispersion relations on the same operators*; it does not show that an abstract Hilbert space cannot carry unrelated representations of both symmetry groups.

At the Lie-algebra level, Galilean boosts commute,

$$[K_i, K_j] = 0,$$

while Lorentz boosts (with a standard normalization) satisfy

$$[K_i, K_j] = -\frac{i\hbar}{c^2}\epsilon_{ijk}J_k.$$

The nonrelativistic symmetry is obtained as a contraction of the relativistic one as $c \to \infty$ (with care over central extensions and normalization).

9 Comparative summary

Feature	Standard NRQM	Local relativistic QFT
Symmetry	Galilei group	Poincaré group
Time	Preferred global parameter	Relativistic spacetime structure
Energy	$\mathbf{p}^2/(2m)$ for free particles	Relativistic energy–momentum spectrum
Locality	No light-cone microcausality postulate	Spacelike commutation of local observables
Particles	Often fixed number	Creation and annihilation allowed
Spin–statistics	Assignment added separately	Theorem under suitable hypotheses
CPT	Not generally implied	Theorem under suitable hypotheses

10 Further questions

From a foundational perspective, promising questions include:

1. What are the *weakest* hypotheses sufficient for CPT and spin–statistics, in either the Wightman or algebraic settings?

2. Which assumptions are independent? Can explicit countermodels be built after removing positivity, locality, or the spectrum condition?

3. To what extent can local field theory be reconstructed from a net of observables and its superselection sectors?

4. What is the exact relation between nonrelativistic limits, group contraction, and the persistence of particle statistics?

These questions distinguish logical entailment inside a theory from empirical reasons for applying a theory to nature.

References and further reading

1. R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Princeton University Press. The classic axiomatic treatment of the two theorems.

2. R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Springer. A standard reference for algebraic QFT.

3. S. Weinberg, *The Quantum Theory of Fields*, Vol. I, Cambridge University Press. Poincaré representations and the structure of relativistic quantum fields.

4. D. Guido and R. Longo, "An algebraic spin and statistics theorem," *Communications in Mathematical Physics* **172** (1995), 517–533.

5. R. Jost, *The General Theory of Quantized Fields*, American Mathematical Society. Analytic methods and CPT.

6. J. C. Baez, "Spin, Statistics, CPT and All That Jazz," https://math.ucr.edu/home/baez/spin_stat.html.