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Explain QM Relativity Incompatibility

You

In what ways is the non-relativistic QM incompatible with Relativity?

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Non-relativistic quantum mechanics (NRQM), as formulated by Schrödinger, is incompatible with **special relativity** in several fundamental ways. The most important are its treatment of time and space, its transformation laws, its dispersion relation, and the possibility of superluminal propagation.

There is also a deeper issue: even replacing the Schrödinger equation with a relativistic wave equation does not completely solve the problem. A fully relativistic quantum theory naturally leads to **quantum field theory (QFT)**.

Let's examine the main incompatibilities.

1. Galilean invariance versus Lorentz invariance

The Schrödinger equation for a free particle is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi.$$

This equation is compatible with Galilean transformations,

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t,$$

provided the wavefunction transforms with an appropriate phase factor.

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Special relativity, however, requires invariance under Lorentz transformations. For a boost along the x-axis,

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right),$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The Schrödinger equation does not retain its form under these transformations.

This is the most direct mathematical incompatibility: NRQM respects the Galilean symmetry group rather than the Poincaré group of special relativity.

2. Absolute time versus relativity of simultaneity

In ordinary QM, the wavefunction

$$\psi(\mathbf{x}, t)$$

describes the quantum state throughout space at a common time t.

Time is an external parameter, whereas position is represented by an operator.

In special relativity, however, simultaneity depends on the observer. Two spatially separated events that are simultaneous in one inertial frame need not be simultaneous in another.

This creates a conceptual tension, especially for entangled systems.

For example, consider two particles in an entangled state,

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$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right).$$

In the non-relativistic formulation, their joint state is defined on a common time slice.

Special relativity does not provide a preferred universal slicing of spacetime.

This does not mean that entanglement violates relativity. Relativistic quantum theories can describe entanglement consistently, but their formulation must accommodate different spacelike hypersurfaces and preserve causal predictions.

3. Non-relativistic energy–momentum relation

NRQM assumes the classical kinetic-energy relation

$$E = \frac{p^2}{2m}.$$

Special relativity requires

$$E^2 = p^2c^2 + m^2c^4,$$

or, for positive-energy particles,

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

Expanding for $p \ll mc$,

$$E = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots.$$

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Thus the Schrödinger Hamiltonian is a low-momentum approximation to the relativistic one, apart from the constant rest energy.

The discrepancy becomes important when the particle's velocity approaches the speed of light.

One can attempt to correct the Schrödinger equation by replacing its Hamiltonian with

$$H = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}.$$

Although this produces the correct free-particle energy–momentum relation, it does not by itself give a fully satisfactory relativistic quantum theory. In particular, the positive-energy Hamiltonian is nonlocal in position space.

4. Instantaneous propagation of wavefunctions

This is one of the most striking differences.

Consider the free Schrödinger equation in one dimension,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}.$$

Its propagator is

$$K(x, t; x', 0) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{i m (x - x')^2}{2\hbar t} \right), \qquad t > 0.$$

Notice that

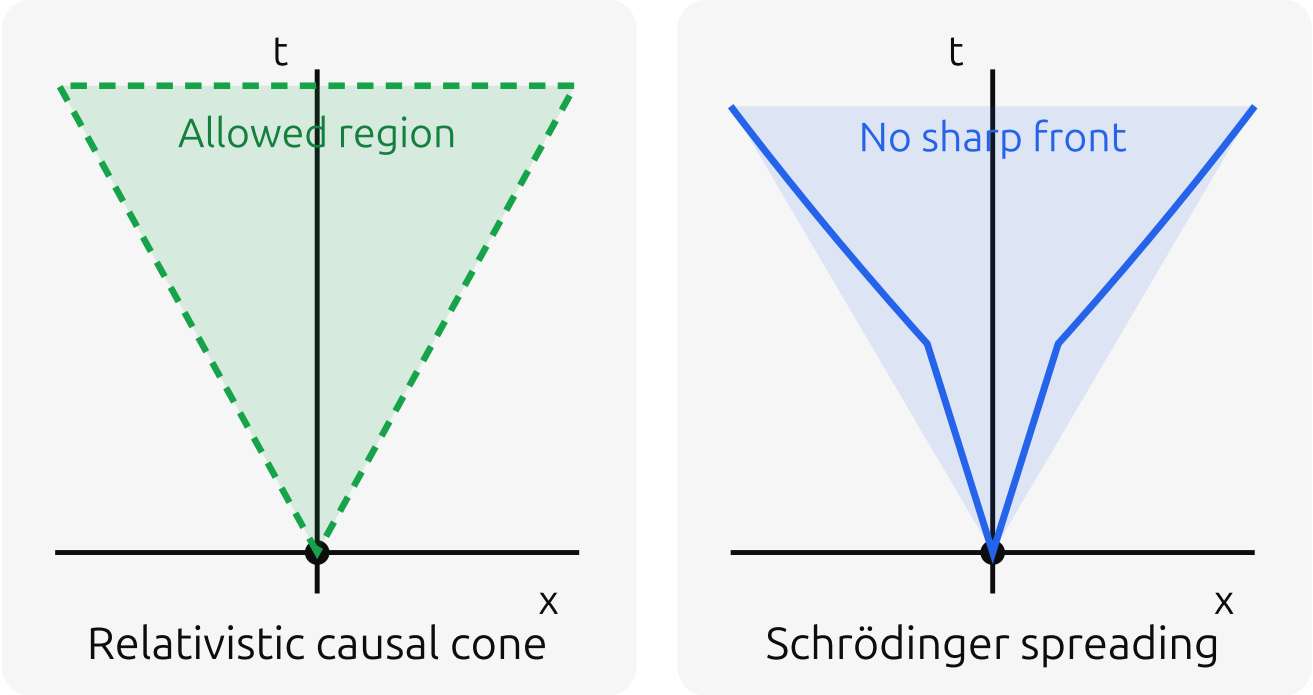
$$K(x, t; x', 0) \neq 0$$

for every finite separation $x - x'$, however large, and every $t > 0$.

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Consequently, a wavefunction initially confined to a finite spatial region immediately develops nonzero amplitudes arbitrarily far away.

Schematic comparison of propagation



Conceptual spacetime diagram: relativistic causal influences remain within the light cone, while Schrödinger evolution has no strict finite propagation speed.

Relativity demands that physical signals cannot propagate faster than c.

The instantaneous spreading in NRQM therefore reveals the absence of a relativistic causal structure.

There is an important qualification: **nonzero probability amplitude outside a light cone is not automatically equivalent to faster-than-light communication**. In relativistic quantum theory, causality is expressed more precisely through the behavior of local observables.

In QFT, for example, local bosonic observables commute at spacelike separation:

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$$[\mathcal{O}(x), \mathcal{O}(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0.$$

This property, called *microcausality*, prevents operations at spacelike separation from being used to transmit signals.

5. Fixed particle number versus particle creation

Ordinary single-particle QM assumes that the particle continues to exist as one particle.

More generally, non-relativistic many-particle QM usually describes systems with a fixed number N of particles.

But special relativity permits the conversion of energy into particles.

For example,

$$\gamma + \gamma \longrightarrow e^- + e^+,$$

provided the kinematic requirements are satisfied.

At sufficiently high energies, particles can be created and destroyed. A theory restricted to a fixed number of particles cannot describe these processes.

Quantum field theory addresses this by introducing a Fock space,

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots,$$

where $\mathcal{H}_n$ represents states containing n particles.

The total state can involve sectors with different particle numbers.

This is a major conceptual shift: **the fundamental quantum objects are fields, rather than a fixed collection of particles.**

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6. Relativistic wave equations do not completely solve the problem

Historically, attempts to reconcile quantum mechanics and relativity led to the Klein–Gordon and Dirac equations.

The Klein–Gordon equation is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi = 0.$$

It is Lorentz invariant, but its standard conserved density is not positive definite, so it cannot generally be interpreted as an ordinary single-particle probability density.

The Dirac equation,

$$(i\hbar c\gamma^\mu \partial_\mu - mc^2)\psi = 0,$$

provides a Lorentz-covariant equation for spin-$\frac{1}{2}$ particles, with a positive probability density for its usual first-quantized interpretation.

However, it includes negative-energy solutions, which point toward antiparticles and the need for a quantum field interpretation when interactions and particle creation are considered.

Both equations become more naturally understood in QFT.

7. A subtle point: the measurement postulate

There is one further issue, more conceptual than the previous ones.

In textbook QM, a measurement is often described as an instantaneous collapse of the wavefunction.

For spatially separated entangled systems, this appears to involve an instantaneous change extending across space.

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But *instantaneous relative to which inertial frame?*

Relativity provides no preferred notion of simultaneity.

Fortunately, the standard predictions of quantum theory do not permit superluminal signalling through entanglement. Different relativistic formulations and interpretations account for the measurement process in different ways.

Thus the tension here is primarily with the textbook description of wavefunction collapse, not with the experimentally verified statistical predictions of quantum mechanics.

Summary

Feature	Non-relativistic QM	Special relativity requires
Symmetry	Galilean invariance	Lorentz/Poincaré invariance
Time	Absolute, universal parameter	Frame-dependent simultaneity
Energy	$p^2/2m$	$E^2 = p^2c^2 + m^2c^4$
Propagation	No finite propagation speed	Causal influences bounded by c
Particle number	Typically fixed	Creation and annihilation allowed
Measurement	Often instantaneous collapse	Frame-independent observable predictions

One final distinction

It would be misleading to say that non-relativistic quantum mechanics and special relativity are logically contradictory theories.

Rather, NRQM is an **approximation with a limited domain of validity**, much like Newtonian mechanics.

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Indeed, NRQM can be recovered from relativistic quantum theory in the appropriate low-energy, low-velocity limit.

The deeper distinction is this:

- Special relativity can be incorporated into quantum mechanics without abandoning quantum principles.
- But a fully consistent relativistic description of interacting quantum particles generally requires quantum field theory.

An especially interesting question is **whether the failure of Lorentz invariance alone already forces us to introduce quantum fields, or whether a genuinely relativistic single-particle quantum mechanics is possible**.

The answer is nuanced: relativistic single-particle theories do exist, particularly for free particles, but locality, interactions, and particle creation make the field-theoretic framework essential for a general theory.

You

Can NR-QM and R-QM be formulated as axiomatic systems?

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Yes. Both non-relativistic quantum mechanics (NRQM) and relativistic quantum mechanics (RQM) can be formulated as mathematical axiomatic systems. In fact, there are several well-developed axiomatizations, differing in their mathematical foundations and in what they regard as fundamental.

An important distinction is that *relativistic quantum mechanics* can mean two different things:

1. **Relativistic single-particle quantum mechanics**, in which quantum states transform consistently under the Poincaré group.
2. **Relativistic quantum field theory (RQFT)**, which allows particle creation and annihilation and incorporates relativistic locality.



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Both can be axiomatized, but the second provides a more comprehensive framework for interacting relativistic systems.

There is also a useful logical distinction: an axiomatization of a physical theory is not necessarily a formal system of first-order logic with recursively enumerable axioms. Often, physicists use mathematical postulates expressed in the language of Hilbert spaces, operator theory, and differential geometry.

1. An axiomatic formulation of NRQM

A standard formulation, influenced by von Neumann and Dirac, can be organized into the following postulates.

Axiom 1. States

Associated with every quantum system is a complex separable Hilbert space $\mathcal{H}$.

Pure physical states correspond to rays in $\mathcal{H}$, or equivalently to rank-one orthogonal projections.

More generally, mixed states are represented by positive trace-class operators ρ satisfying

$$\rho \geq 0, \qquad \operatorname{Tr} \rho = 1.$$

Axiom 2. Observables

Physical observables are represented by self-adjoint operators on $\mathcal{H}$.

If A is an observable, its possible measurement outcomes are described by its spectral measure P_A.

Axiom 3. Probabilities (Born rule)

For a system in state ρ, the probability that a measurement of A yields a value in a Borel set Δ is

$$\Pr(A \in \Delta) = \operatorname{Tr}(\rho P_A(\Delta)).$$

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Axiom 4. Time evolution

Between measurements, an isolated system evolves through a strongly continuous one-parameter group of unitary operators,

$$U(t) = e^{-itH/\hbar},$$

where H is a self-adjoint Hamiltonian.

For pure states, this gives the Schrödinger equation,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

Axiom 5. Composition

The Hilbert space of a composite system is the tensor product

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

For identical particles, states are restricted to the symmetric or antisymmetric subspaces, as appropriate for bosons or fermions.

Axiom 6. Spacetime symmetry

Non-relativistic systems transform covariantly under the Galilei group, with quantum states carrying projective unitary representations of that group.

For a massive spinless particle, this leads to the familiar kinematic structure involving

$$[X_i, P_j] = i\hbar\delta_{ij} I,$$

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and, for a free particle,

$$H = \frac{\mathbf{P}^2}{2m}$$

up to an additive constant.

The first five postulates are largely general quantum principles. The sixth distinguishes the non-relativistic theory from its relativistic counterpart.

This distinction is important: **the Hilbert-space axioms of quantum theory are not intrinsically non-relativistic.** The symmetry, dynamics, and locality requirements determine the spacetime character of the theory.

2. An axiomatic formulation of relativistic single-particle QM

We can retain the first several axioms of NRQM, but replace the Galilean symmetry requirement with Poincaré symmetry.

The Poincaré group consists of spacetime translations and Lorentz transformations:

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes O(1,3).$$

For most purposes we begin with its proper orthochronous subgroup.

Axiom R1. States and probabilities

States are represented by rays in a complex Hilbert space $\mathcal{H}$, with probabilities given by the Born rule.

Axiom R2. Poincaré covariance

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The Hilbert space carries a strongly continuous projective unitary representation of the proper orthochronous Poincaré group, or equivalently an ordinary unitary representation of an appropriate covering group:

$$U(a, \Lambda).$$

Transformations act consistently on physical states and observables.

Axiom R3. Positive energy

The generators of translations satisfy the relativistic spectrum condition:

$$\operatorname{spec}(P^\mu) \subseteq \overline{V}_+,$$

where $\overline{V}_+$ is the closed forward light cone in energy–momentum space.

In particular,

$$H \geq 0.$$

Axiom R4. Elementary particles

An elementary particle is associated with an irreducible positive-energy unitary representation of the Poincaré group.

For a massive particle, the invariants include

$$P_\mu P^\mu = m^2 c^2$$

and its spin s, determined by the Pauli–Lubanski operator.

These axioms lead naturally to **Wigner's classification of elementary particles**.

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For a massive particle, the possible states are characterized by mass, spin, and momentum, and satisfy

$$E^2 = p^2c^2 + m^2c^4.$$

This is already a genuine relativistic quantum theory at the single-particle level.

However, these axioms do not yet give a sufficiently general theory of local interactions.

In particular, they do not automatically provide a satisfactory notion of strictly localized particles compatible with relativistic causality.

3. Axiomatic relativistic quantum field theory

Two particularly important frameworks are:

- **Wightman axioms**, formulated using quantum fields as operator-valued distributions.
- **Haag–Kastler axioms**, formulated using algebras of local observables.

The Haag–Kastler approach is especially interesting because locality is expressed directly as an algebraic requirement.

The Haag–Kastler framework

Instead of starting with particles, assign to every suitable bounded spacetime region O an algebra of observables,

$$O \longmapsto \mathcal{A}(O).$$

One can formulate the following axioms.

Axiom Q1. Isotony

If one spacetime region is contained in another,

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$$O_1 \subseteq O_2,$$

then

$$\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2).$$

An observable localized in a smaller region is also available in any larger region containing it.

Axiom Q2. Locality (microcausality)

If O_1 and O_2 are spacelike separated, then

$$[A, B] = 0$$

for every

$$A \in \mathcal{A}(O_1), \qquad B \in \mathcal{A}(O_2).$$

This is the mathematical expression of compatibility with relativistic causal structure.

Axiom Q3. Poincaré covariance

There is an action of the Poincaré group such that

$$U(g)\mathcal{A}(O)U(g)^{-1} = \mathcal{A}(gO).$$

The observable algebra transforms consistently with spacetime transformations.

Axiom Q4. Spectrum condition

The joint spectrum of energy and momentum lies in the forward light cone:

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$$\mathrm{spec}(P^\mu) \subseteq \overline{V}_+.$$

Axiom Q5. Vacuum

In the vacuum representation, there is a normalized Poincaré-invariant vacuum vector Ω:

$$U(g)\Omega = \Omega.$$

Uniqueness of the vacuum is often imposed as an additional condition.

These axioms, together with appropriate technical assumptions, provide a rigorous framework for relativistic quantum field theory.

Notice that no Schrödinger equation or particular Lagrangian appears among these axioms.

The axioms are structural. They define a class of possible physical theories rather than one particular theory.

4. The two axiomatic systems compared

Property	NRQM	Relativistic QFT
Mathematical foundation	Hilbert spaces and operators	Hilbert spaces and local operator algebras
Spacetime symmetry	Galilei group	Poincaré group
Time evolution	Unitary evolution	Unitary evolution
Probabilities	Born rule	Born rule
Causality	No relativistic locality axiom	Spacelike commutativity
Energy	Bounded below	Relativistic spectrum condition

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Particle number	Usually fixed in standard models	May vary
Fundamental locality	Spatial observables at fixed time	Observable algebras assigned to spacetime regions

An important observation is that the two theories share most of their general quantum structure.

The primary differences are in the **spacetime symmetry and locality axioms**.

5. Can these axioms be expressed in formal logic?

Yes, although there are several levels of formalization.

At the first level, the postulates are expressed in ordinary mathematical language, as above.

At a second level, one can formalize them within a foundational mathematical theory such as ZFC set theory.

Hilbert spaces, self-adjoint operators, Lie groups, operator algebras, and their representations can all be defined within ZFC.

Consequently, one could formulate NRQM and RQFT as theories with appropriately specified axioms in the language of set theory.

At a third level, one can investigate these theories using mathematical logic, asking questions about:

- **Consistency:** Can the axioms be satisfied simultaneously?
- **Independence:** Can particular axioms be derived from the others?
- **Existence:** Are there mathematical models satisfying the axioms?
- **Uniqueness:** To what extent do the axioms characterize a particular theory?
- **Logical strength:** What mathematical foundations are required to establish the results?

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For example, free relativistic quantum fields furnish models of the standard axioms. Certain interacting quantum field theories have also been constructed rigorously in lower spacetime dimensions.

However, establishing the existence of nontrivial interacting quantum field theories in four spacetime dimensions is much more difficult. This is closely related to the mathematical challenges behind the Yang–Mills existence and mass gap problem.

6. Can one derive incompatibility between NRQM and relativity from their axioms?

This is perhaps the most interesting consequence of taking an axiomatic approach.

Suppose we require one quantum system to satisfy both:

1. The standard Galilean transformation laws, including absolute time and the nonrelativistic boost structure.
2. The Lorentz transformation laws of special relativity, with the same physical energy–momentum operators.

For a massive free particle, the non-relativistic representation gives

$$H = \frac{\mathbf{P}^2}{2m} + E_0,$$

whereas relativistic kinematics requires

$$H^2 = c^2\mathbf{P}^2 + m^2c^4.$$

These conditions cannot both hold for a particle whose momentum takes arbitrary values.

Indeed, substituting the first into the second gives

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$$\left(\frac{\mathbf{P}^2}{2m} + E_0\right)^2 = c^2\mathbf{P}^2 + m^2c^4.$$

This is a polynomial identity in $\mathbf{P}^2$ that cannot hold over a continuum of momentum values, because its left side contains a term proportional to $\mathbf{P}^4$, while its right side does not.

So, under these physical assumptions, the combined kinematic requirements are inconsistent.

There is also a more abstract approach using Lie algebras.

The Galilei algebra contains boost generators K_i satisfying

$$[K_i, K_j] = 0,$$

whereas the corresponding Lorentz boost generators satisfy, in a conventional normalization,

$$[K_i, K_j] = -\frac{i\hbar}{c^2}\epsilon_{ijk}J_k.$$

Thus the two symmetry structures differ at the algebraic level.

The Galilei algebra can be obtained as a contraction of the Poincaré algebra in the formal limit

$$c \longrightarrow \infty.$$

This explains mathematically why the non-relativistic theory emerges as an approximation to the relativistic theory, rather than as an exact alternative description valid at all velocities.

One qualification is important: merely requiring a Hilbert space to carry representations of both groups is not contradictory. The incompatibility arises when the representations are required to describe the **same physical spacetime transformations and dynamical quantities**.

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7. A deeper axiomatic question

There is an intriguing distinction between two projects:

Project A: Axiomatize quantum mechanics and then impose either Galilean or Poincaré symmetry.

Project B: Find a minimal collection of quantum and spacetime principles from which the appropriate dynamical framework can be derived.

Project B is considerably more ambitious.

For instance, Wigner's theorem relates transformations preserving quantum transition probabilities to unitary or antiunitary operators. Stone's theorem relates continuous unitary time evolution to self-adjoint generators. Representation theory then strongly constrains the possible forms of particle dynamics.

But these principles alone do not uniquely determine a physical quantum theory. Additional assumptions about interactions, locality, particles, and observables remain necessary.

My conclusion: It is entirely possible to treat NRQM and RQM as axiomatic mathematical theories and investigate their relationship rigorously. Moreover, the distinction between the Galilei and Poincaré groups provides a particularly clean mathematical setting for doing so.

One especially worthwhile next step would be to formulate a **precise incompatibility theorem**: identify a small collection of non-relativistic quantum axioms and relativistic axioms whose union has no nontrivial model, and give a rigorous proof. This would distinguish genuine logical inconsistency from the weaker statement that NRQM is merely an approximation to RQM.

You

How about proofs for theorems such as CPT and the spin–statistics theorem? Do they fit in one or more of the axiomatic frameworks you mentioned?

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Yes. In fact, the CPT theorem and the spin–statistics theorem are two of the most important achievements of axiomatic quantum field theory. They are rigorous mathematical consequences of appropriate sets of axioms, rather than additional physical postulates that must be assumed independently.

Both the **Wightman framework** and the **Haag–Kastler (algebraic) framework** admit proofs of these results, although the hypotheses and mathematical techniques differ.

This is precisely why axiomatic QFT is valuable: it allows us to identify which general physical principles are sufficient to establish fundamental properties of quantum particles.

1. Which axiomatic systems support the theorems?

Framework	CPT theorem	Spin–statistics theorem
Non-relativistic QM	Not generally derivable	Not generally derivable
Relativistic single-particle QM	Not generally derivable	Not generally derivable
Wightman axiomatic QFT	Yes	Yes
Haag–Kastler algebraic QFT	Yes, with suitable additional hypotheses	Yes, with suitable additional hypotheses

The distinction is important.

In non-relativistic QM, we can describe bosons and fermions, but their association with integer and half-integer spin is not forced by the usual axioms.

For example, the mathematical framework of NRQM permits us to construct a model of spin-zero fermions or spin-$\frac{1}{2}$ bosons. Such models would not have the standard physical interpretation as local relativistic particles in four-dimensional spacetime, but they are not inherently inconsistent

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as non-relativistic quantum models.

It is the combination of **relativistic covariance, locality, positive energy, and quantum positivity** that makes the spin–statistics connection a theorem.

Similarly, Lorentz covariance alone does not imply CPT invariance. We need additional assumptions involving locality and the structure of the quantum theory.

These distinctions are discussed in the classic monograph by Streater and Wightman, PCT, Spin and Statistics, and All That. Princeton U… +1

2. The CPT theorem in Wightman's axiomatic framework

Consider a quantum field theory in four-dimensional Minkowski spacetime satisfying the standard Wightman axioms.

The principal hypotheses include:

1. A positive-definite Hilbert space with quantum fields defined as operator-valued distributions.
2. A unitary representation of the proper orthochronous Poincaré group, with covariantly transforming fields.
3. A Poincaré-invariant vacuum vector.
4. The spectrum condition: energy–momentum lies in the closed forward light cone.
5. Local commutativity or graded local commutativity at spacelike separation, with the appropriate assumptions on the fields.
6. The usual regularity, domain, and vacuum cyclicity conditions.

Under these assumptions, one obtains the CPT theorem.

The theorem

There exists an antiunitary operator Θ, representing the combined CPT transformation, such that

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$$\Theta U(a,\Lambda)\Theta^{-1} = U(-a,\Lambda)$$

for proper orthochronous Lorentz transformations Λ, in a standard convention.

Its action on fields involves spacetime inversion, adjunction or charge conjugation, and the appropriate spin-dependent matrices.

For example, for a Hermitian scalar field, with a suitable phase convention,

$$\Theta\phi(x)\Theta^{-1} = \phi(-x).$$

For charged and spinorial fields the transformation law is more elaborate.

The theorem does **not** assert that C, P, or T are separately symmetries. They may each be violated, while their combined CPT transformation remains a symmetry.

How the proof works

One of the most elegant features of the proof is its use of complex analysis.

Define the vacuum correlation functions

$$W_n(x_1,\dots,x_n) = \langle \Omega, \phi(x_1)\cdots\phi(x_n)\Omega\rangle.$$

The proof proceeds, schematically, through three stages.

Step 1: Analyticity from the spectrum condition.

The restriction on energy–momentum implies that the Wightman functions are boundary values of analytic functions in appropriate domains of complexified Minkowski spacetime.

Step 2: Analytic continuation of Lorentz covariance.

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Lorentz covariance extends to suitable complex Lorentz transformations.

Crucially, total spacetime inversion,

$$x^\mu \longmapsto -x^\mu,$$

is accessible through the connected complex Lorentz group, even though it is not connected to the identity in the real proper orthochronous Lorentz group.

Step 3: Locality and reversal of field order.

Locality allows appropriate field products to be reordered at spacelike separations.

Combining this with the analytic continuation establishes identities among the Wightman functions that express CPT symmetry.

A reconstruction argument then yields the antiunitary CPT operator on the Hilbert space.

The role of locality can actually be weakened: Jost's CPT theorem characterizes CPT symmetry in terms of *weak local commutativity* at suitable configurations called Jost points. QFT.org +1

This is a genuinely axiomatic proof. It does not require specifying a particular Lagrangian or solving the equations of motion of a particular field theory.

3. The spin–statistics theorem in Wightman's framework

The spin–statistics theorem establishes the familiar connection

$$\boxed{\begin{aligned} \text{Integer spin} &\Longrightarrow \text{Bose statistics},\\ \text{Half-integer spin} &\Longrightarrow \text{Fermi statistics}. \end{aligned}}$$

In four-dimensional local relativistic QFT, under the standard assumptions, the alternatives are not freely interchangeable.

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A mathematical formulation

Let $\phi(x)$ be a nonzero local quantum field transforming according to an appropriate finite-dimensional representation of the Lorentz group.

At spacelike separation, the exchange relations have the form

$$\phi_a(x)\phi_b(y) = \eta\,\phi_b(y)\phi_a(x),$$

in a simplified single-statistics setting, where

$$\eta = \begin{cases} +1 & \text{Bose commutation,} \\ -1 & \text{Fermi anticommutation.} \end{cases}$$

The spin–statistics theorem implies

$$\boxed{\eta = (-1)^{2s}}$$

for a field associated with spin s, subject to the usual qualifications for fields with several spin components and charged fields.

A more precise formulation says that imposing the wrong spacelike exchange relations on a field, while retaining the relevant Wightman assumptions, forces that field to be trivial.

Thus a nontrivial spin-zero field cannot obey the fermionic exchange relations under these assumptions, and a nontrivial spin-$\frac{1}{2}$ field cannot obey bosonic ones.

Why is relativity essential?

The proof again exploits a combination of properties unavailable in NRQM:

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- The representation theory of the Lorentz group, which distinguishes integer and half-integer spin.
- Analyticity properties of vacuum correlation functions derived from the spectrum condition.
- Locality, which constrains the exchange of spacelike-separated fields.
- Positivity of the Hilbert-space inner product.

In particular, complex Lorentz transformations and locality lead to constraints on correlation functions, while Hilbert-space positivity rules out nontrivial fields with the wrong spin–statistics assignment.

It is worth emphasizing that **Lorentz invariance by itself does not prove the theorem**. Nor does the elementary observation that two exchanges restore the original ordering.

For an accessible discussion of the ideas behind the rigorous proof, see John Baez's Spin, Statistics, CPT and All That Jazz ↗. math.ucr.edu

4. Can these theorems also be proved in Haag–Kastler algebraic QFT?

Yes, and this is particularly interesting because the algebraic approach starts from observables, not from a prescribed collection of quantum fields.

Recall that a Haag–Kastler theory assigns an observable algebra $\mathcal{A}(O)$ to each spacetime region O.

One of its basic requirements is locality:

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$$

when O_1 and O_2 are spacelike separated.

At first glance, this condition appears to describe only commuting, bosonic observables. How, then, can fermionic statistics emerge?

The answer involves **superselection sectors**.

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Different representations of the observable algebra can describe different kinds of charges and particles. In the Doplicher–Haag–Roberts (DHR) theory, suitable localized superselection sectors possess an intrinsic statistics structure.

In four spacetime dimensions, under the appropriate localization and finiteness assumptions, that structure distinguishes Bose and Fermi statistics. It can be related to the spin determined by the corresponding representation of spacetime rotations.

The Guido–Longo theorem

A particularly relevant result is the 1995 paper by Daniele Guido and Roberto Longo, *An algebraic spin and statistics theorem*.

They prove both spin–statistics and PCT results in an algebraic setting, under additional assumptions involving the geometric action of modular groups associated with wedge-region von Neumann algebras.

Their central assumption is closely connected to the Bisognano–Wichmann property, which identifies certain modular transformations with Lorentz boosts. doi Springer Nat… +1

This is an important qualification: **the minimal Haag–Kastler axioms, as I summarized them earlier, are not by themselves sufficient to establish every version of CPT and spin–statistics**. The rigorous algebraic theorems specify additional structural hypotheses.

The algebraic approach therefore offers a different route to the same broad physical conclusions, without making particular local field operators the starting point.

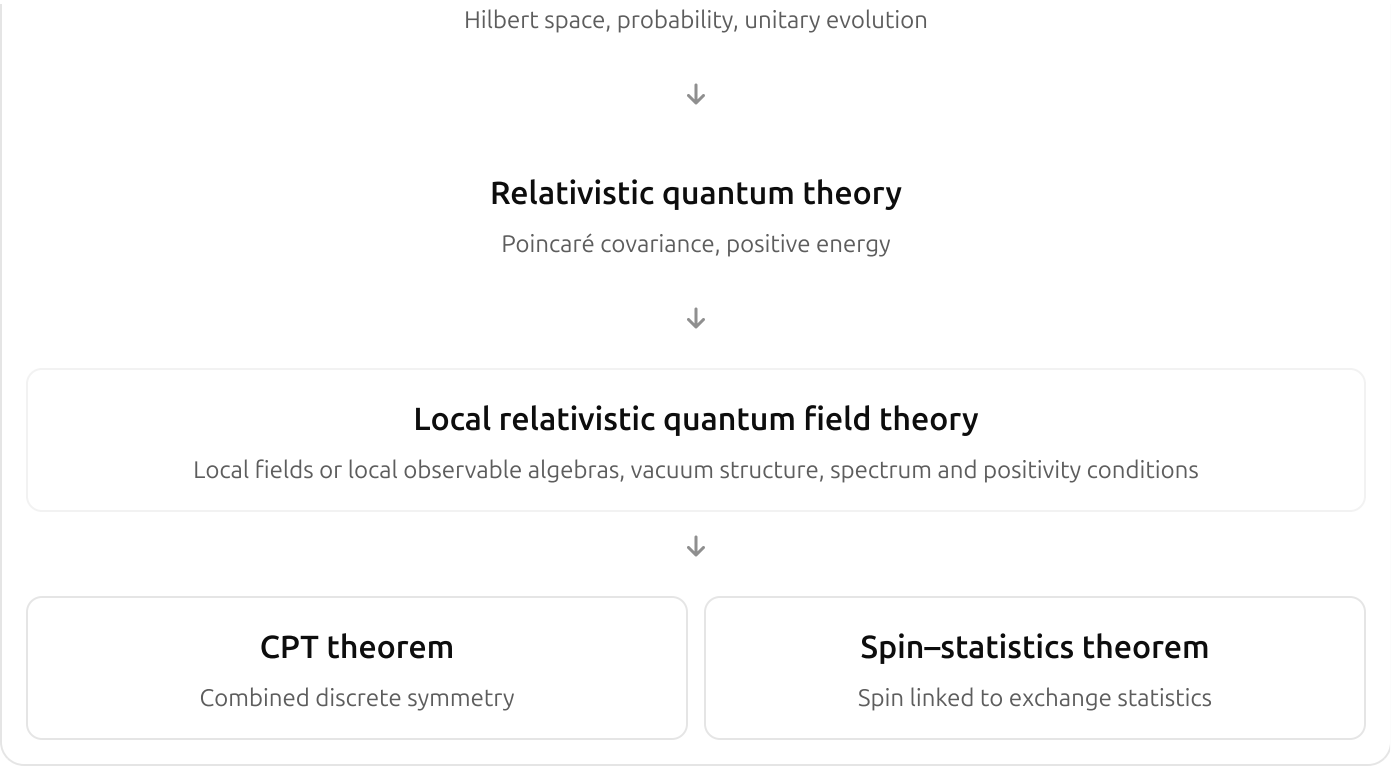
5. Are these theorems purely consequences of special relativity?

No. This is an important logical distinction.

Consider the following hierarchy:

Quantum mechanics

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This diagram represents a sufficient chain of assumptions, not an assertion that each box is a uniquely determined extension of the previous one.

The two theorems are not consequences of the abstract Hilbert-space formalism alone, or even of Poincaré symmetry alone.

Moreover, some hypotheses matter in subtle ways. For example, positivity of the physical inner product is important in spin–statistics results. Gauge-fixed field formulations sometimes use indefinite-metric auxiliary spaces in which unphysical ghost fields can have unconventional spin–statistics assignments without contradicting the theorem about physical states.

Spacetime dimension also matters. In $2+1$ dimensions, more general exchange statistics, including anyonic statistics, are possible. The familiar Bose–Fermi dichotomy is specifically associated with the standard setting in three spatial dimensions.

6. What is the logical significance?

This brings us back to your original question about axiomatic systems.

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Suppose W denotes a suitable formalization of the Wightman axioms. Then, schematically, we have

$$W \vdash \mathrm{CPT}$$

and

$$W \vdash \mathrm{SpinStatistics}.$$

In other words, CPT and spin–statistics can be established as theorems within the mathematical theory specified by the axioms.

However, this leaves several logically distinct questions:

Consistency. Does there exist a model of the axioms? Free quantum fields provide nontrivial examples, so the consistency of the usual framework is supported relative to the underlying mathematical foundations.

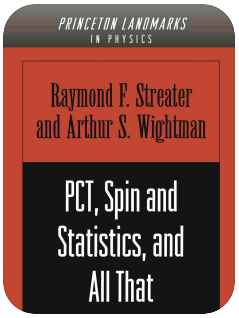
Independence. Which assumptions are indispensable? For instance, what happens if we weaken locality or positivity? This is an active and mathematically meaningful question, with different answers depending on the precise theorem.

Completeness. Do the axioms determine a unique quantum field theory? Certainly not. Many different models can satisfy the same structural axioms.

Physical applicability. Does a particular model of the axioms adequately describe nature? This requires experiment, and it is distinct from proving that certain properties hold in every model satisfying the axioms.

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Recommended reference



PCT, Spin and Statistics, and All That

Raymond F. Streater and Arthur S. Wightman.

This is perhaps the most directly relevant reference for your question. It develops rigorous axiomatic QFT and proves fundamental results, including the CPT and spin–statistics theorems.

Princeton University Press ↗

A particularly interesting next question would be: **Can we isolate a minimal set of axioms sufficient to prove CPT or spin–statistics, and construct counterexamples when individual axioms are omitted?**

That would turn the discussion into a mathematical investigation of the precise logical dependencies of two fundamental physical theorems.

You

Can you write this whole discussion as an editable LaTeX document for my records?

Worked for 2m 44s >

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I've prepared a self-contained **8-page LaTeX article** consolidating our discussion, with equations, mathematical formulations, explanations, and references.

Nonrelativistic and Relativistic Quantum Mechanics: Incompatibilities, Axiomatic Frameworks, CPT, and Spin–Statistics

Download the editable LaTeX source (.tex) ↗

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The document covers:

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1. The incompatibilities between NRQM and special relativity.
2. Axiomatic formulations of NRQM and relativistic single-particle QM.
3. The Wightman and Haag–Kastler frameworks.
4. The CPT theorem and the mathematical structure of its proof.
5. The spin–statistics theorem and its essential hypotheses.
6. The algebraic approach through superselection sectors.
7. Questions of consistency, independence, and formal logical derivability.
8. An illustrative incompatibility argument and references for further study.

I've also incorporated some technical refinements, particularly concerning the precise hypotheses of the CPT and spin–statistics theorems.

The LaTeX source is fully editable, and I verified that it compiles successfully with `pdflatex`.

NRQM_RQM_Axioms_CPT_SpinStatistics.tex
File

NRQM_RQM_Axioms_CPT_SpinStatistics.pdf
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