SOME APPLICATIONS OF EXTREMAL FUNCTIONS IN FOURIER ANALYSIS

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ABSTRACT. This is a preliminary report about a work that is still in progress. We begin with a brief introduction to some extremal functions in Fourier analysis, such as Beurling's and Selberg's, and others obtained from solving certain problems of interpolation of given functions at points of the form $n+\delta$ for some fix $-\frac{1}{2} < \delta \leq \frac{1}{2}$. Then we present some new results and conjectures.

1. Introduction

One subject of interest in the theory of uniform distribution modulo one is the estimation of the discrepancy of a sequence $x_1, x_2, \ldots, x_M$ by an expression depending on trigonometric sums of the form $\sum_{m=1}^{M} e^{2\pi i n x_m}$. The discrepancy gives a measure of how much a given sequence gets apart from uniform distribution, and is defined as follows:

$$\Delta^*(x_1, \ldots, x_M) = \Delta^*_M = \sup_{s<t<s+1} \left| \sum_{m=1}^{M} \chi_{s,t}(x_m) - M(t-s) \right| \tag{1.1}$$

where

$$\chi_{s,t}(x) = \begin{cases} 1 & \text{if } s < x - n < t \ \text{ for some } n \in \mathbb{Z} \\ \frac{1}{2} & \text{if } s - x \in \mathbb{Z} \ \text{ or } t - x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

It is interesting to note that an estimation of the discrepancy could have practical applications, such as the estimation of the error of algorithms for numerical integration, particularly those based in computing means of values of the integrand at points of the interval of integration. This is accomplished by the Koksma's inequality: Let f be a function

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on the interval $[0,1]$ of bounded variation $V(f)$, and suppose we are given M points in $[0,1]$ with discrepancy Δ_M^*. Then:

$$\left| \sum_{n=1}^{M} f(x_n) - M \int_0^1 f(t)\,dt \right| \;\le\; V(f)\,\Delta_M^* \tag{1.3}$$

One well known upper bound for the discrepancy is given by the Erdös-Turán inequality ([1]), which has the form

$$\Delta_M^* \;\le\; c_1 \frac{M}{N} \;+\; c_2 \sum_{n=1}^{N} \frac{1}{n} \left| \sum_{m=1}^{M} e^{2\pi i n x_m} \right| \tag{1.4}$$

Where c_1 and c_2 are positive constants and N is an integer that can be chosen so as to minimize the right hand side of (1.4).

That result has been refined by Vaaler ([4]), using a slightly different definition for the discrepancy:

$$\Delta(x_1,\dots,x_M) \;=\; \Delta_M \;=\; \sup_{y\in\mathbb{R}} \left| \sum_{m=1}^{M} \psi(y - x_m) \right| \tag{1.5}$$

where

$$\psi(x) \;=\; \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases} \tag{1.6}$$

It can be verified that $\Delta_M \le \Delta_M^* \le 2\Delta_M$, so both discrepancies differ insignificantly.

The refinement was made by using some extremal functions, as discussed in [3]. A short exposition of the main results follows.

2. Some Extremal Functions in Fourier Analysis

In the late 1930's A. Beurling observed that the entire function

$$B(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} \;-\; \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2} \;+\; \frac{2}{z} \right\} \tag{2.1}$$

satisfies the following properties:

1. Is entire of exponential type 2π.

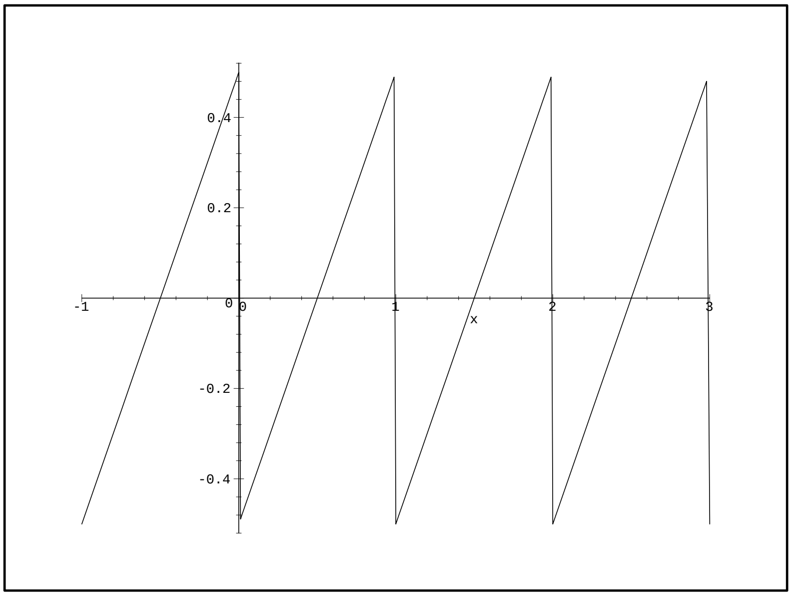

FIGURE 1. The function $\psi(x)$

2. Majorizes $\operatorname{sgn}(x)$ (the sign of x) along the real axis:

$$\operatorname{sgn}(x) \;\leq\; B(x) \qquad \text{for every } x \in \mathbb{R}$$

3. Verifies:

$$\int_{-\infty}^{\infty} \{\, B(x) - \operatorname{sgn}(x) \,\} \, dx \;=\; 1 \tag{2.2}$$

4. Is extremal, in the sense that among all functions satisfying 1 and 2, it is the one that minimizes integral (2.2) in 3.

In 1974 Selberg used Beurling's function $B(z)$ to obtain a sharp form of the large sieve inequality. He also noted that the function

$$C_E \;=\; \frac{1}{2}\,\{B(\beta - z) + B(z - \alpha)\} \tag{2.3}$$

majorizes the characteristic function χ_E of the interval $E = [\alpha, \beta]$ (see [2]):

$$\chi_E(x) \;=\; \frac{1}{2}\,\{\operatorname{sgn}(\beta - z) + \operatorname{sgn}(z - \alpha)\} \;\leq\; C_E(x) \tag{2.4}$$

Beurling's function is a particular case of a more general class of functions. If $F(z)$ is an entire function of exponential type 2π, bounded

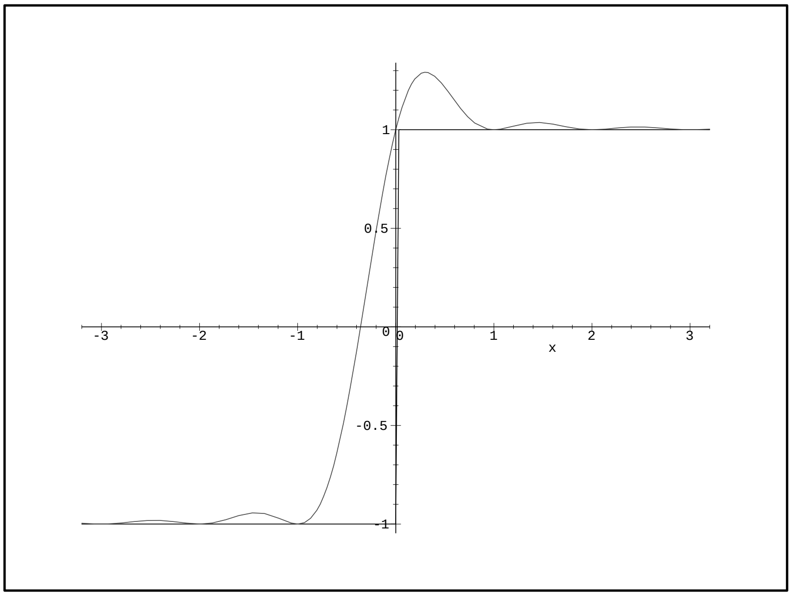

FIGURE 2. Beurling's function $B(x)$ majorizing $\text{sgn}(x)$

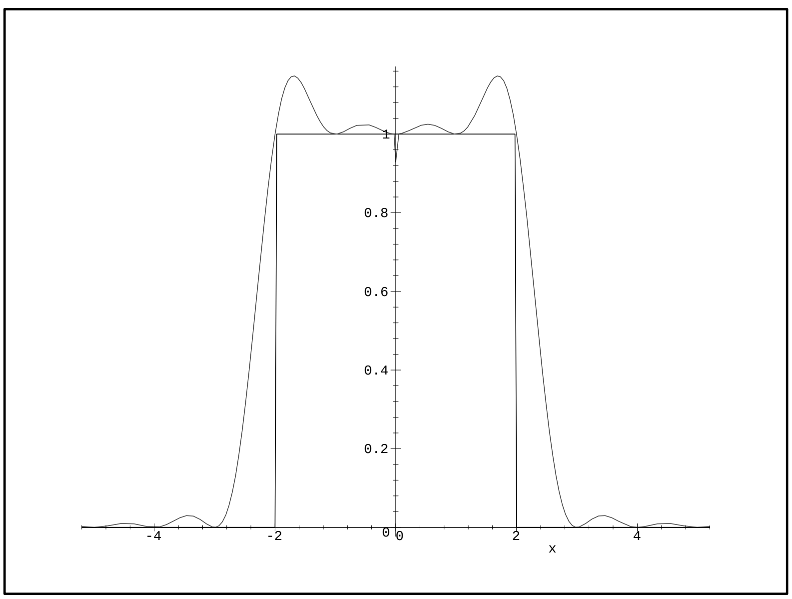

FIGURE 3. Selberg's function $C(x)$ majorizing an interval

on $\mathbb{R}$, and odd, then it can be represented by the interpolation formula:

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{F(n)}{(z-n)^2} + \sum_{n=-\infty}^{\infty} \frac{F'(n)}{z-n} \right\} \tag{2.5}$$

So, by giving suitable values to the numerators of the terms of that series it is possible to "force" F and its derivative to take prescribed values at the integers.

Note that $B(z)$ can be interpreted as a function that majorizes $\operatorname{sgn}(x)$, and interpolates that function and its derivative at the non zero integers, i.e., $B(n) = \operatorname{sgn}(n)$ and $B'(n) = \operatorname{sgn}'(n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$. Since we are interested in a majorizing function, $B(0)$ must be 1. On the other hand $\operatorname{sgn}(x)$ has no derivative at zero, so $B'(0)$ can be left as a parameter to be determined later. It turns out that the "right" value for $B'(0)$ is precisely 2.

Modifying the definition of $B(z)$ by giving it a value of zero at zero yields a slightly different (and more symmetric) function (note that $\operatorname{sgn}(0) = 0$):

$$H(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} + \frac{2}{z} \right\} \tag{2.6}$$

Now $H(z)$ interpolates $\operatorname{sgn}(x)$ at the integers and its derivative at the non zero integers, but it is not a majorizing function of $\operatorname{sgn}(x)$ any more. However we recover $B(z)$ just by adding the following function:

$$K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 \tag{2.7}$$

i.e.:

$$B(z) = H(z) + K(z) \tag{2.8}$$

By *subtracting* $K(z)$ from $H(z)$ we get

$$-B(-z) = H(z) - K(z) \tag{2.9}$$

which has the property of *minorizing* $\operatorname{sgn}(x)$ along the real axis:

$$-B(-x) \le \operatorname{sgn}(x) \qquad \text{for every } x \in \mathbb{R} \tag{2.10}$$

The facts that $B(z)$ is an extremal function in the sense of minimizing integral (2.2), and is also an interpolating function for $\operatorname{sgn}(x)$ at the integers, are connected. In fact, if $F(z)$ is some other function with the

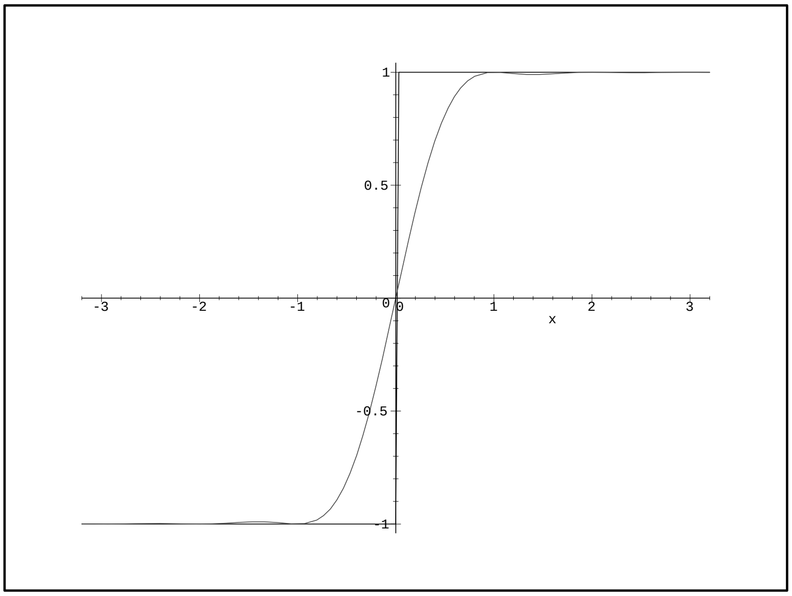

FIGURE 4. Function $H(x)$

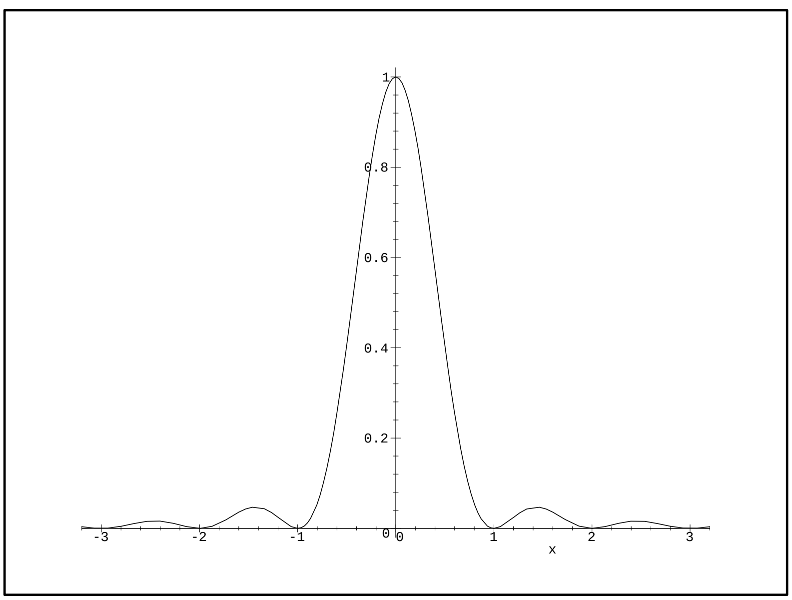

FIGURE 5. Function $K(x)$

same properties 1 and 2 as $B(z)$, writing $D(x) = F(x) - \mathrm{sgn}(x)$ and using the Poisson summation formula we get:

$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \sum_{m=-\infty}^{\infty} \hat{D}(m)\, e^{2\pi i m x} \tag{2.11}$$

where

$$\hat{D}(t) \;=\; \int_{-\infty}^{\infty} D(x)\, e^{-2\pi i t x}\, dx \tag{2.12}$$

is the Fourier transform of $D(x)$.

It can be proven that the Fourier transform of $F'(x)$ is supported in $[-1, 1]$, and this implies that for $|t| \geq 1$, $\hat{D}(t)$ equals the Fourier transform of $B(x) - \mathrm{sgn}(x)$, which is $-\frac{1}{\pi i t}$ (for $|t| \geq 1$), hence:

$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \hat{D}(0) \;-\; \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{\pi i m}\, e^{2\pi i m x} \;=\; \hat{D}(0) + 2\,\psi(x) \tag{2.13}$$

where $\psi(x)$ is the function defined in (1.6). Note that $\psi(0^+) = -\frac{1}{2}$, hence:

$$\sum_{l=-\infty}^{\infty} D(l^+) \;=\; \hat{D}(0) \;-1 \tag{2.14}$$

Since $D(x)$ is non negative, we get that:

$$\int_{-\infty}^{\infty} D(x)\, dx \;=\; \hat{D}(0) \;\geq\; 1 \tag{2.15}$$

so that B is in fact extremal. Furthermore, if we want F to be extremal, then we need $\hat{D}(0) = 1$, but this implies:

$$\sum_{l=-\infty}^{\infty} D(l^+) \;=\; 0 \tag{2.16}$$

so that $D(l^+) = 0$, i.e., $F(l^+) = \mathrm{sgn}(l^+)$ for every integer l. Since $F(x) \geq \mathrm{sgn}(x)$, also $F'(l) = 0$ for every non zero integer l (the derivative at zero can be determined by a slightly more refined argument). Hence F is an interpolating function for $\mathrm{sgn}(x)$ at the integers. Actually, by using the expansion (2.5) we get that F is precisely B, i.e., B is the only function verifying properties 1–3.

A couple of examples of application of the properties of B are shown next.

Theorem 2.1 (Montgomery and Vaughan)**.** *Let* $\lambda_1, \lambda_2, \dots, \lambda_N$ *be real numbers satisfying* $|\lambda_m - \lambda_n| \geq \delta > 0$ *whenever* $m \neq n$*, and let* $a(1), \dots, a(N)$ *be arbitrary complex numbers. Then*

$$\left| \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{\lambda_m - \lambda_n} \right| \;\leq\; \frac{\pi}{\delta} \sum_{n=1}^{N} |a(n)|^2 \tag{2.17}$$

Proof. We may assume $\delta = 1$. The idea of the proof is to use $D(x) = B(x) - \operatorname{sgn}(x)$, which we know is non negative, and $\hat{D}(0) = 1$, $\hat{D}(t) = -\frac{1}{\pi i t}$ for $|t| \geq 1$. Then we use

$$\begin{aligned} 0 \;&\leq\; \int_{-\infty}^{\infty} D(x) \left| \sum_{n=1}^{N} a(n)\, e^{-2\pi i \lambda_n x} \right|^2 \\ &=\; \sum_{m=1}^{N} \sum_{n=1}^{N} a(m)\,\overline{a(n)}\,\hat{D}(\lambda_m - \lambda_n) \qquad (2.18) \\ &=\; \hat{D}(0) \sum_{n=1}^{N} |a(n)|^2 \;+\; \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} a(m)\,\overline{a(n)}\,\hat{D}(\lambda_m - \lambda_n) \end{aligned}$$

By putting the terms with $m = n$ at one side of the inequality and those with $m \neq n$ at the other side we get:

$$\sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{i\,(\lambda_m - \lambda_n)} \;\leq\; \pi \sum_{n=1}^{N} |a(n)|^2 \tag{2.19}$$

By using $-B(-x) - \operatorname{sgn}(x)$ instead of $B(x) - \operatorname{sgn}(x)$, we get the inequality reversed and a factor of -1 in the right side. From here the result follows. □

Theorem 2.2 (Erdös-Turán inequality)**.** *If* $x_1, x_2, \dots, x_M$ *are real numbers and* N *a positive integer, then*

$$\Delta_M \;\leq\; \frac{M}{2N} \;+\; \left\{1 + \frac{1}{\pi}\right\} \sum_{n=1}^{N} \frac{1}{n} \left| \sum_{m=1}^{M} e^{2\pi i n x_m} \right| \tag{2.20}$$

Hence the Erdös-Turán inequality (1.4) holds with $c_1 = \frac{1}{2}$ and $c_2 = 1 + \frac{1}{\pi}$.

3. A MAJORIZING EXTREMAL FUNCTION FOR THE LOGARITHM

A way to study a sequence $x_1, x_2, \ldots$ modulo 1 is by looking at the sequence $e^{2\pi i x_m}$ on the unit circle. An analogous of the concept of "discrepancy" of M points $x_1, x_2, \ldots, x_M$ can be obtained by considering the following expression:

$$\Gamma_M = \frac{1}{\pi} \log \sup_{y\in\mathbb{R}} \left| \prod_{m=1}^{M} (e^{2\pi i y} - e^{2\pi i x_n}) \right| = \sup_{y\in\mathbb{R}} \sum_{m=1}^{M} \varphi(y - x_m) \tag{3.1}$$

where

$$\varphi(x) = \begin{cases} \frac{1}{\pi} \log|2\,\sin \pi x| & \text{if } x \in \mathbb{R}\setminus\mathbb{Z} \\ -\infty & \text{if } x \in \mathbb{Z} \end{cases} \tag{3.2}$$

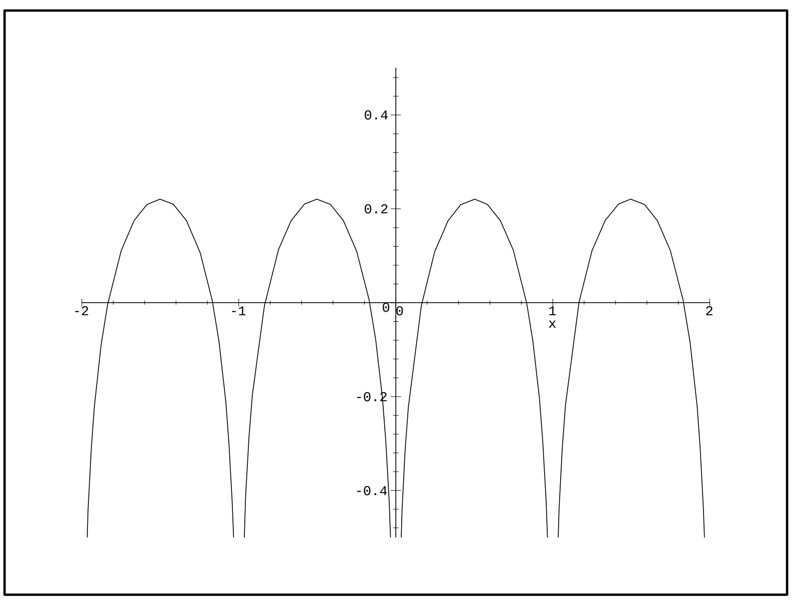

FIGURE 6. The function $\varphi(x)$

Note the similarity with the definition (1.5) of the discrepancy Δ_M. Also note that ψ and φ are (Fourier) conjugate functions, as can be deduced from their Fourier expansions[1]:

$$\hat{\psi}(n) \;=\; \begin{cases} -\frac{1}{2\pi i n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{3.3}$$

and

$$\hat{\varphi}(n) \;=\; \begin{cases} -\frac{1}{2\pi |n|} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{3.4}$$

so that

$$\psi(x) \;=\; -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n x) \tag{3.5}$$

and

$$\varphi(x) \;=\; -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos(2\pi n x) \tag{3.6}$$

Another interesting remark is that Γ_M can be interpreted as $\frac{1}{\pi}$ times the supremum in the unit disk of the absolute value of a polynomial whose roots lie all on the unit circle:

$$\Gamma_M \;=\; \Gamma(x_1, x_2, \dots, x_M) \;=\; \frac{1}{\pi} \sup \{|P_M(z)| : |z| \leq 1\} \tag{3.7}$$

where

$$P_M(z) \;=\; \prod_{m=1}^{M} (z - e^{2\pi i x_m}) \tag{3.8}$$

Recall that $\psi(x)$ appeared naturally in 2.13 after applying the Poisson summation formula to $\sum_{l=-\infty}^{\infty} \{B(x+l) - \operatorname{sgn}(x+l)\}$. This justifies to pose an analogous problem for the the conjugate function of $\operatorname{sgn}(x)$, namely $\frac{2}{\pi} \log |x|$. If we get an entire function $F(z)$ of exponential type 2π that majorizes $\log |x|$ along the real axis and proceed analogously to the Beurling's function, then we get:

(3.9)

$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \hat{D}(0) \;+\; \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{2|m|} e^{2\pi i m x} \;=\; \hat{D}(0) - \pi\, \varphi(x)$$

now with $D(x) = F(x) - \log |x|$. Since $D(x) \geq 0$ then

$$\hat{D}(0) \geq \pi\, \varphi(x) \tag{3.10}$$

[1]See [5] for the relation between Fourier and harmonic conjugate functions.

The maximum of $\varphi(x)$ is $\frac{1}{\pi}\log 2$, attained at $n + 1/2$ ($n \in \mathbb{Z}$). If we want F to be extremal in the same sense as the Beurling's function, then:

$$\int_{-\infty}^{\infty} D(x)\,dx \;=\; \hat{D}(0) \;=\; \log 2 \tag{3.11}$$

which implies $D(\frac{1}{2} + l) = 0$ for every $l \in \mathbb{Z}$. This implies that $F(x)$ must interpolate $\log|x|$ and its derivative at the integers plus a half, so its expansion must be:

(3.12)

$$F(z) \;=\; \Big(\frac{\cos \pi z}{\pi}\Big)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\log|n+\frac{1}{2}|}{(z-(n+\frac{1}{2}))^2} \;+\; \sum_{n=-\infty}^{\infty} \frac{\frac{1}{n+\frac{1}{2}}}{z-(n+\frac{1}{2})} \right\}$$

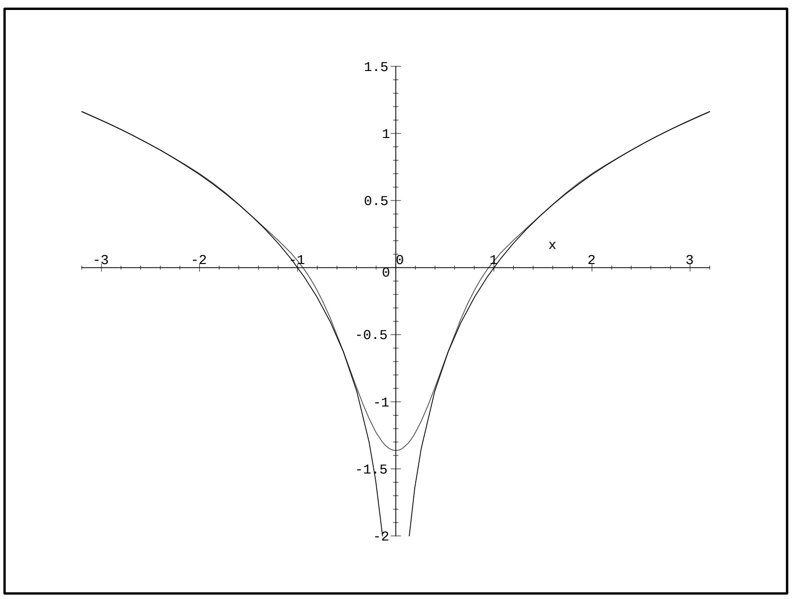

FIGURE 7. Function $F(x)$ majorizing $\log(|x|)$

It turns out that this function is in fact a majorizing function for $\log|x|$:

$$\log|x| \;\leq\; F(x) \qquad \text{for every } x \in \mathbb{R} \tag{3.13}$$

From here analogous results to those obtained with the Beurling's function follow:

Theorem 3.1 (Analogous of Montgomery and Vaughan's inequality)**.** *Let* $\lambda_1, \lambda_2, \dots, \lambda_N$ *be real numbers satisfying* $|\lambda_m - \lambda_n| \geq \delta > 0$ *whenever* $m \neq n$*, and let* $a(1), \dots, a(N)$ *be arbitrary complex numbers. Then*

$$-\frac{2\,\log 2}{\delta} \sum_{n=1}^{N} |a(n)|^2 \;\leq\; \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{|\lambda_m - \lambda_n|} \tag{3.14}$$

Proof. The proof is similar to that of theorem 2.1, but using $D(x) = F(x) - \log|x|$ instead of $B(x) - \operatorname{sgn}(x)$. □

It can be proven that the constant $2\log 2$ is best possible. Note that we only get a lower bound, since the form in the right hand side is known to be unbounded above.

Theorem 3.2 (Erdös-Turán-type inequality)**.** *If* $x_1, x_2, \dots, x_M$ *are real numbers and* N *a positive integer, then*

$$\Gamma_M \;\leq\; \frac{M\,\log 2}{\pi\,(N+1)} \;+\; \frac{1}{\pi} \sum_{n=1}^{N} \frac{1}{n} \left| \sum_{m=1}^{M} e^{2\pi i n x_m} \right| \tag{3.15}$$

Proof. This result can be obtained by applying the Poisson summation formula to the following expression, where $D(x) = F(x) - \log|x|$ (≥ 0):

$$\begin{aligned} \sum_{l=-\infty}^{\infty} D((N+1)\,(x+l)) \;&=\; \frac{1}{N+1} \sum_{n=-\infty}^{\infty} \hat{D}\Big(\frac{n}{N+1}\Big)\, e^{2\pi i n x} \\ &=\; \frac{\log 2}{N+1} \;-\; \sum_{n=1}^{N} \left\{ \frac{1}{n} - \frac{2\,\hat{D}(\frac{n}{N+1})}{N+1} \right\} \cos 2\pi n x \;-\; \pi\,\varphi(x) \end{aligned} \tag{3.16}$$

From here we get the following bound for $\varphi(x)$:

$$\varphi(x) \;\leq\; \frac{\log 2}{\pi\,(N+1)} \;-\; \frac{1}{\pi} \sum_{n=1}^{N} \left\{ \frac{1}{n} - \frac{2\,\hat{D}(\frac{n}{N+1})}{N+1} \right\} \cos 2\pi n x \tag{3.17}$$

Writing the definition of Γ_M, using

$$-\sum_{m=1}^{M} \cos 2\pi n(y - x_m) \;\leq\; \left| \sum_{m=1}^{M} e^{2\pi i n(y - x_m)} \right| \;=\; \left| \sum_{m=1}^{M} e^{2\pi i n x_m} \right| \tag{3.18}$$

and taking into account that $0 \leq \hat{D}(t) \leq \frac{1}{2|t|}$, we get the announced result. □

4. Future work

Remark: Some results in this section, in spite of having some evidence in their support, are still pendent for rigurous proof.

4.1. Generalization of the majorizing function for $\log|x|$.

A small variation of the problems studied above would be to substitute $\log|x|$ with $\frac{1}{2}\log(w^2+x^2)$ $(w>0)$. Note that $\log|x| = \lim_{w\to 0^+}\frac{1}{2}\log(w^2+x^2)$. Then equation 3.9 becomes:

(4.1)
$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \hat{D}(0) \;+\; \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{e^{-2\pi w|m|}}{2|m|}\, e^{2\pi i m x} \;=\; \hat{D}(0) - \pi\,\varphi_w(x)$$

where

(4.2)
$$\begin{aligned}\varphi_w(x) &= -w + \frac{1}{2\pi}\log(2\cosh 2\pi w - 2\cos 2\pi x)\\ &= \frac{1}{\pi}\log\left|\, e^{2\pi i x} - e^{-2\pi w}\,\right|\end{aligned}$$

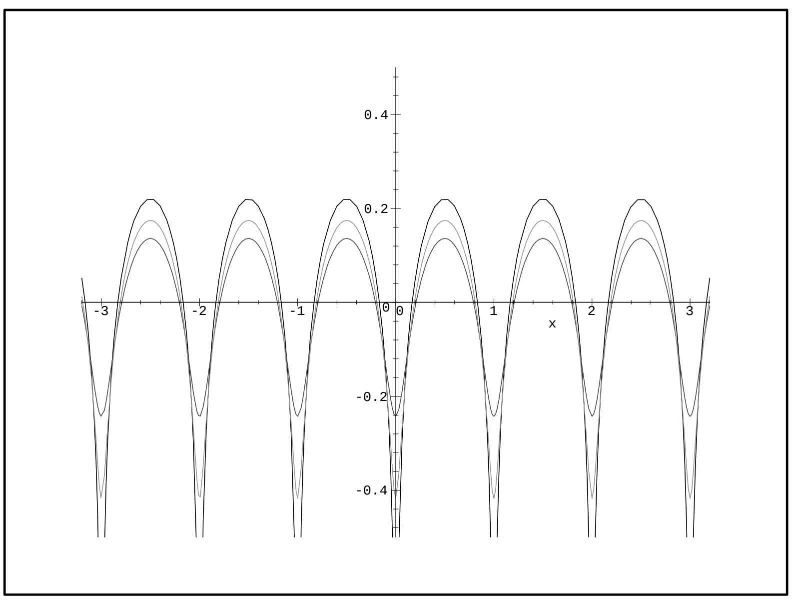

FIGURE 8. The function $\varphi_w(x)$ for several values of w

That function becomes $\varphi_(x)$ for $w \to 0^+$, but has some extra properties of interest. It is periodic of period 2π and has maxima and minima respectively at the points of the form $n + \frac{1}{2}$ and n for $n \in \mathbb{Z}$, where it takes the values:

$$\begin{aligned} \varphi_w(n + \tfrac{1}{2}) &= \frac{1}{\pi} \log\left(1 + e^{-2\pi w}\right) \\ \varphi_w(n) &= \frac{1}{\pi} \log\left(1 - e^{-2\pi w}\right) \end{aligned} \tag{4.3}$$

Reasoning as in the previous section we have that the following is a good candidate for an entire function of exponential type 2π that majorizes $\frac{1}{2} \log(w^2 + x^2)$ and is extremal in the sense of minimizing the L^1 norm of $F_w^+(x) - \frac{1}{2} \log(w^2 + x^2)$:

$$F_w^+(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\frac{1}{2} \log\{w^2 + (n + \frac{1}{2})^2\}}{(z - (n + \frac{1}{2}))^2} + \sum_{n=-\infty}^{\infty} \frac{\frac{n+\frac{1}{2}}{w^2+(n+\frac{1}{2})^2}}{z - (n + \frac{1}{2})} \right\} \tag{4.4}$$

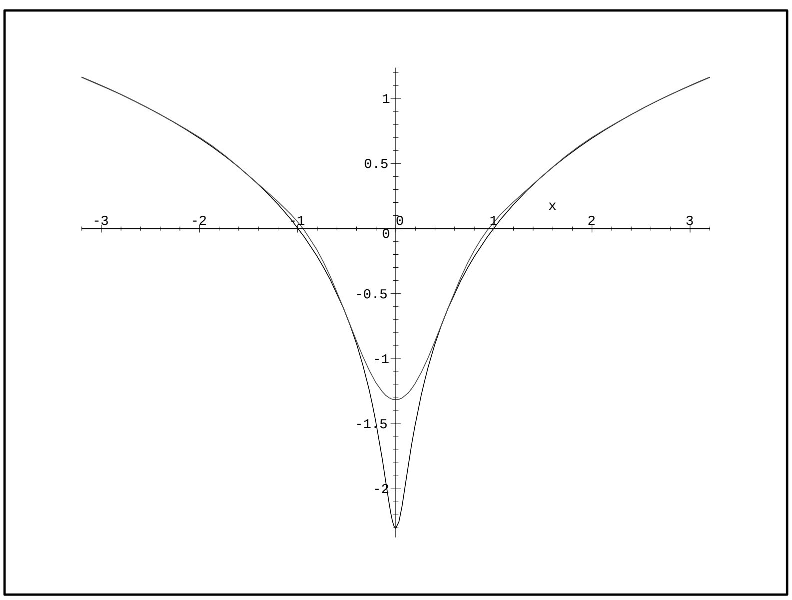

FIGURE 9. Function $F_w^+(x)$ majorizing $\frac{1}{2} \log(w^2 + x^2)$

However now we also have a candidate for a *minorizing* extremal function, which will result from interpolating at the integers instead of the integers plus a half:

(4.5)

$$F_w^-(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\frac{1}{2}\,\log\{w^2+n^2\}}{(z-n)^2} \;+\; \sum_{n=-\infty}^{\infty} \frac{\frac{n}{w^2+n^2}}{z-n} \right\}$$

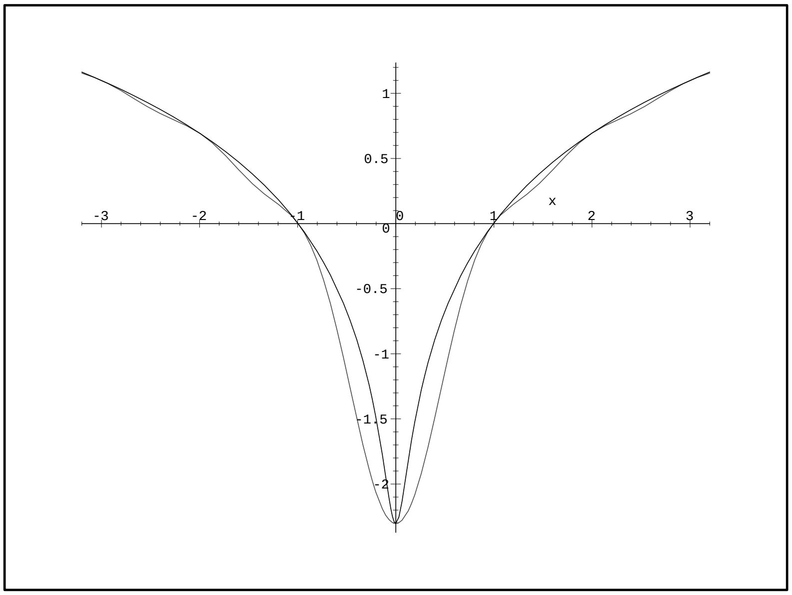

FIGURE 10. Function $F_w^-(x)$ minorizing $\frac{1}{2}\,\log(w^2+x^2)$

If we define $D_w^{\pm}(x) = F_w^{\pm}(x) - \frac{1}{2}\,\log(w^2+x^2)$, then:

(4.6)

$$\begin{aligned} \hat{D}_w^+(0) \;&=\; \log\left(1+e^{-2\pi w}\right) \\ \hat{D}_w^-(0) \;&=\; \log\left(1-e^{-2\pi w}\right) \end{aligned}$$

If it is confirmed that $F_w^-(x) \le \frac{1}{2}\,\log(w^2+x^2) \le F_w^+(x)$ for every $x \in \mathbb{R}$, then several results would follow, such as:

Conjecture 4.1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \ge \delta > 0$ whenever $m \ne n$, and let $a(1), \ldots, a(N)$ be arbitrary* *complex numbers. Also let* α *be any positive real number. Then*

$$(4.7)\quad -\frac{2}{\delta}\,\log\left(1+e^{-2\pi\alpha\delta}\right)\sum_{n=1}^{N}|a(n)|^2$$

$$\le \sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{n=1}^{N}\frac{a(m)\,\overline{a(n)}}{|\lambda_m-\lambda_n|}\,e^{-2\pi\alpha|\lambda_m-\lambda_n|}$$

$$\le -\frac{2}{\delta}\,\log\left(1-e^{-2\pi\alpha\delta}\right)\sum_{n=1}^{N}|a(n)|^2$$

In the proof for this result we will set $w=\alpha\,\delta$, use (4.6) and

$$(4.8)\qquad \hat{D}_w^{\pm}(t) \;=\; \frac{e^{-2\pi w|t|}}{2|t|}\qquad \text{for } |t|\ge 1$$

and proceed as in the proof of theorem 2.1.

Note that this result becomes theorem 3.1 for $\alpha\to 0^+$. Note also that the upper bound tends to infinity.

4.2. **Generalization of Beurling's function.** Note that $\frac{1}{2}\log(w^2+x^2)$ is just $\log|w+x\,i|$, so its harmonic conjugate will be $\arg(w+x\,i)=\arctan\frac{x}{w}$. This function becomes $\frac{\pi}{2}\,\mathrm{sgn}(x)$ for $w\to 0^+$. If we substitute $\mathrm{sgn}(x)$ by $\frac{2}{\pi}\arctan\frac{x}{w}$ and repeat the work done for the Beurling's function, we get:

$$(4.9)\qquad \sum_{l=-\infty}^{\infty} D(x+l) \;=\; \hat{D}(0) \;-\; \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{-2\pi w|m|}}{\pi i m}\,e^{2\pi i m x} \;=\; \hat{D}(0)+2\,\psi_w(x)$$

where

$$(4.10)\qquad \psi_w(x) \;=\; -\frac{1}{\pi}\,\arctan\left\{\frac{\sin 2\pi x}{e^{2\pi w}-\cos 2\pi x}\right\}$$

The function $\psi_w(x)$ becomes $\psi(x)$ for $w\to 0^+$, but also has some properties of interest. It is periodic of period 2π and has maxima and minima respectively at the points of the form $n-\delta_w$ and $n+\delta_w$ for $n\in\mathbb{Z}$, where

$$(4.11)\qquad \delta_w \;=\; \frac{1}{2\pi}\,\arccos\left\{e^{-2\pi w}\right\}\qquad 0<\delta_w<\frac{1}{2}$$

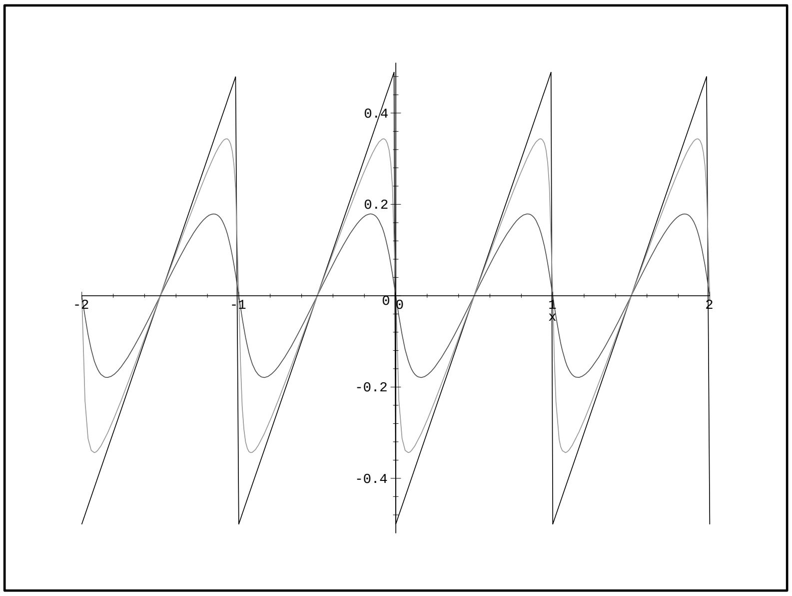

FIGURE 11. The function $\psi_w(x)$ for several values of w

and it takes the values:

$$\psi_w(n \pm \delta_w) \;=\; \mp\frac{1}{\pi}\,\arctan\left\{(e^{4\pi w} - 1)^{-\frac{1}{2}}\right\} \tag{4.12}$$

From here we get that the following function is a candidate for an entire extremal function of exponential type 2π that majorizes $\frac{2}{\pi}$ arctan $\frac{x}{w}$ along the real axis:

$$B_w^+(z) \;=\; \left(\frac{\sin\pi(z-\delta_w)}{\pi}\right)^2\left\{\sum_{n=-\infty}^{\infty}\frac{\frac{2}{\pi}\arctan\frac{n+\delta_w}{w}}{(z-(n+\delta_w))^2} \;+\; \sum_{n=-\infty}^{\infty}\frac{\frac{2w/\pi}{w^2+(n+\delta_w)^2}}{z-(n+\delta_w)}\right\} \tag{4.13}$$

By substituting δ_w with $-\delta_w$ we get an analogous function, $B_w^-(z)$, intended to minorize arctan $\frac{x}{w}$ along the real axis.

Note that by letting $w \to 0^+$, then $\delta_w \to 0^+$, and $B_w^+(z)$ and $B_w^-(z)$ approach $B(z)$ and $-B(-z)$ respectively.

Also, we note that $B_w^{+\prime}(\delta_w) \to 2$ as $w \to 0^+$, which confirms the value $B'(0) = 2$ for the derivative of the Beurling's function at zero.

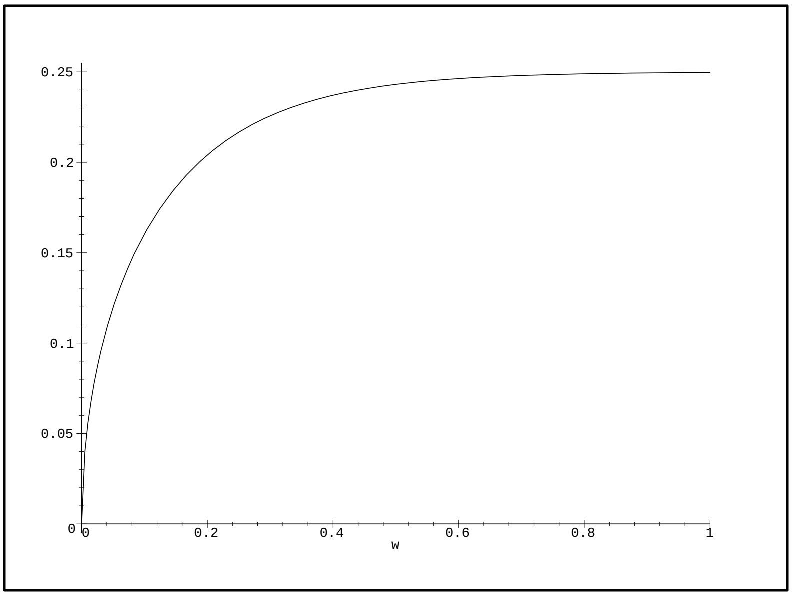

FIGURE 12. δ_w as a function of w

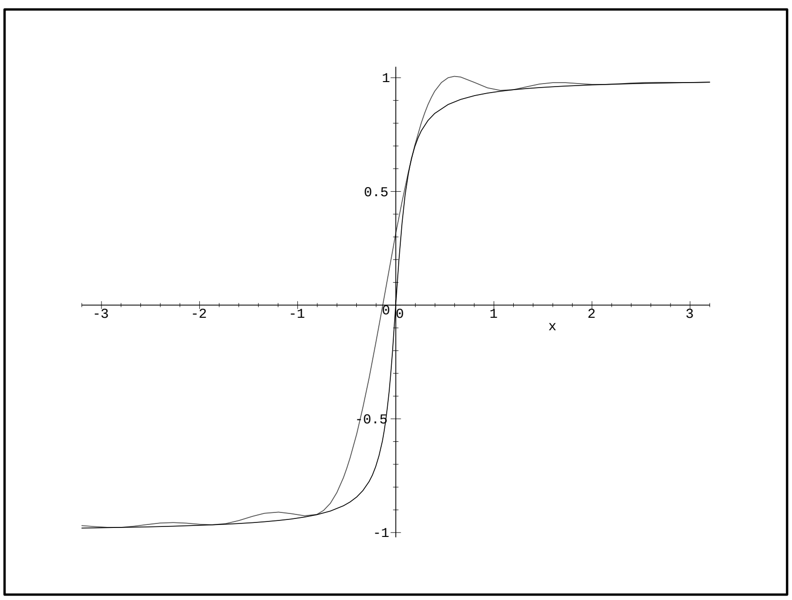

FIGURE 13. Function $B_w^+(x)$ majorizing $\frac{2}{\pi}\arctan\frac{x}{w}$

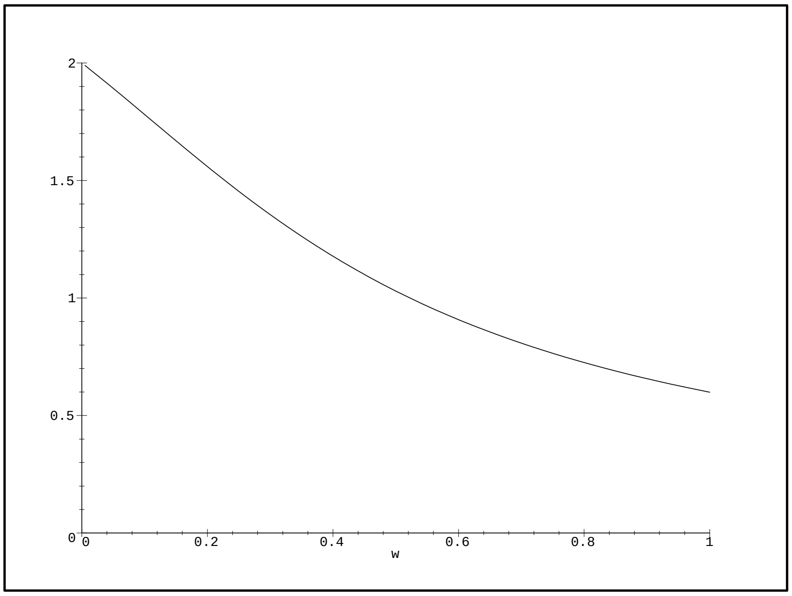

FIGURE 14. ${B_w^{\pm}}'(\delta_w)$ as a function of w

If $D_w^{\pm} = B_w^{\pm}(x) - \frac{2}{\pi} \arctan \frac{x}{w}$ then:

$$\hat{D}_w^{\pm}(0) \;=\; \pm\frac{2}{\pi} \arctan\left\{(e^{4\pi w} - 1)^{-\frac{1}{2}}\right\} \tag{4.14}$$

Next, some results that follow from the extremal properties of $B_w^{\pm}(x)$:

Conjecture 4.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \geq \delta > 0$ whenever $m \neq n$, and let $a(1), \ldots, a(N)$ be arbitrary complex numbers. Also let α be any positive real number. Then*

$$\left| \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{\lambda_m - \lambda_n}\, e^{-2\pi\alpha|\lambda_m - \lambda_n|} \right| \leq \frac{2}{\delta} \arctan\left\{(e^{4\pi\alpha\delta} - 1)^{-\frac{1}{2}}\right\} \sum_{n=1}^{N} |a(n)|^2 \tag{4.15}$$

In the proof of this result we will set $w = \alpha\,\delta$, use (4.14) and

$$D_w^{\pm}(t) \;=\; -\frac{e^{-2\pi w|t|}}{\pi i t} \qquad \text{for } |t| \geq 1 \tag{4.16}$$

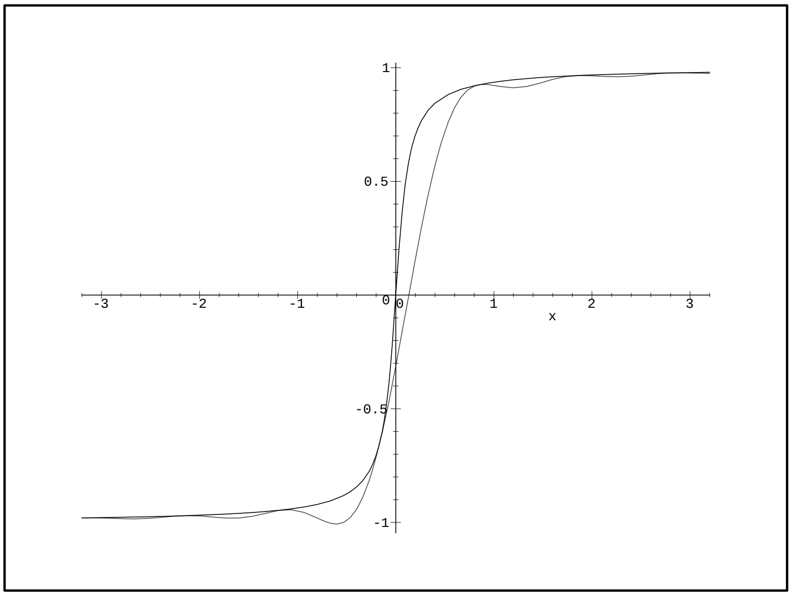

FIGURE 15. Function $B_w^-(x)$ minorizing $\frac{2}{\pi}\arctan\frac{x}{w}$

and proceed as in the proof of theorem 2.1.

By letting $\alpha \to 0^+$ this result becomes theorem 2.1, so 4.15 can be considered as a generalization of Montgomery and Vaughan's inequality.

5. CONCLUSIONS

We have dealt with the problem of finding some entire function $F(z)$ of exponential type 2π that majorizes a given function $f(x)$ along the real axis and is extremal in the sense that the L^1 norm of $F(x) - f(x)$ is minimum. We have looked at some cases in which this problem can be solved by defining F as a function that interpolates $f(x)$ and $f'(x)$ at the points of the form $n + \delta$ for some $-1/2 < \delta \leq 1/2$. The relation can be shown, and the value of δ can be found, by using the Poisson summation formula (2.11).

The function so obtained can be used for simplifying the proofs of some known results, such as the the Montgomery and Vaughan inequality, and the Erdös-Turán bound, and to get improvements and suitable generalizations of those results.

References

[1] Paul Erdös and Paul Turan. On a problem in the theory of uniform distribution, I. *Indag. Math.*, 10:370–378, 1948.

[2] Atle Selberg. *Collected Papers*, volume II. Springer-Verlag, 1991.

[3] Jeffrey D. Vaaler. Some extremal functions in Fourier analysis. *Bulletin of the American Mathematical Society (New Series)*, 12(2):183–216, April 1985.

[4] Jeffrey D. Vaaler. Refinements of the Erdös-Turan inequality. In *Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991)*, number 147 in Lectures Notes in Mathematics, pages 263–269. Dekker, 1993.

[5] Antoni Zygmund. *Trigonometric Series*, volume I and II. Cambridge University Press, second edition, 1968.

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