Notes on
Calculus II
Integral Calculus

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November 25, 2004

Contents

Introduction

These notes are intended to be a summary of the main ideas in course MATH 214-2: *Integral Calculus*. I may keep working on this document as the course goes on, so these notes will not be completely finished until the end of the quarter.

The textbook for this course is Stewart: *Calculus, Concepts and Contexts* (2th ed.), Brooks/Cole. With few exceptions I will follow the notation in the book.

If you find any typos or errors, or you have any suggestions, please, do not hesitate to let me know. You may email me, or use the web form for feedback on the web pages for the course:

`http://www.math.northwestern.edu/~mlerma/courses/math214-2-04f/`

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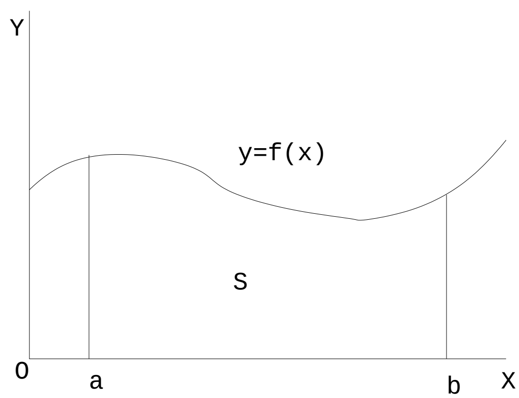
Fall 2004

CHAPTER 1

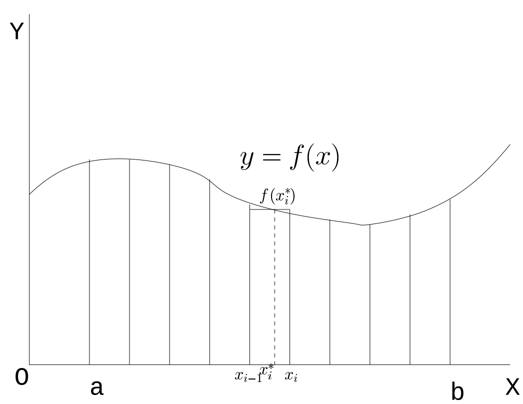
Integrals

1.1. Areas and Distances. The Definite Integral

1.1.1. The Area Problem. The Definite Integral. Here we try to find the area of the region S under the curve $y = f(x)$ from a to b, where f is some continuous function.



In order to estimate that area we begin by dividing the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\dots$, $[x_{n-1}, x_n]$, each of length $\Delta x = (b - a)/n$ (so $x_i = a + i\Delta x$).



The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f(x_i^*)$, where x_i^* is a sample point in the interval $[x_i, x_{i+1}]$. So the total area under the curve is approximately the sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x = f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\,.$$

This expression is called a *Riemann Sum*.

The estimation is better the thiner the strips are, and we can identify the exact area under the graph of f with the limit:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

As long as f is continuous the value of the limit is independent of the sample points x_i^* used.

That limit is represented $\int_a^b f(x)\,dx$, and is called *definite integral* of f from a to b:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The symbols at the left historically were intended to mean an infinite sum, represented by a long "S" (the integral symbol $\int$), of infinitely small amounts $f(x)\,dx$. The symbol dx was interpreted as the length of an "infinitesimal" interval, sort of what Δx becomes for infinite n. This interpretation was later abandoned due to the difficulty of reasoning with infinitesimals, but we keep the notation.

Remark: Note that in intervals where $f(x)$ is negative the graph of $y = f(x)$ lies below the x-axis and the definite integral takes a negative value. In general a definite integral gives the *net* area between the graph of $y = f(x)$ and the x-axis, i.e., the sum of the areas of the regions where $y = f(x)$ is above the x-axis minus the sum of the areas of the regions where $y = f(x)$ is below the x-axis.

1.1.2. Evaluating Integrals. We will soon study simple and efficient methods to evaluate integrals, but here we will look at how to evaluate integrals directly from the definition.

Example: Find the value of the definite integral $\int_0^1 x^2\,dx$ from its definition in terms of Riemann sums.

Answer: We divide the interval $[0,1]$ into n equal parts, so $x_i = i/n$ and $\Delta x = 1/n$. Next we must choose some point x_i^* in each subinterval $[x_{i-1}, x_i]$. Here we will use the right endpoint of the interval $x_i^* = i/n$. Hence the Riemann sum associated to this partition is:

$$\sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 1/n = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \frac{2n^3 + 3n^2 + n}{6} = \frac{2 + 3/n + 1/n^2}{6}\,.$$

So:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{2 + 3/n + 1/n^2}{6} = \frac{1}{3}\,.$$

In order to check that the result does not depend on the sample points used, let's redo the computation using now the left endpoint of each subinterval:

$$\sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 1/n = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{1}{n^3} \frac{2n^3 - 3n^2 + n}{6} = \frac{2 - 3/n + 1/n^2}{6}\,.$$

So:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{2 - 3/n + 1/n^2}{6} = \frac{1}{3}\,.$$

1.1.3. The Midpoint Rule. The *Midpoint Rule* consists of computing Riemann sums using $\overline{x}_i = (x_{i-1} + x_i)/2 =$ midpoint of each interval as sample point. This yields the following approximation for the value of a definite integral:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)]\,.$$

Example: Use the Midpoint Rule with $n = 5$ to approximate $\int_0^1 x^2\,x$.

Answer: The subintervals are $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 1]$, the midpoints are $0.1, 0.3, 0.5, 0.7, 0.9$, and $\Delta x = 1/5$, so

$$\int_0^1 x^2\,dx \approx \frac{1}{5}\,\left[0.1^2 + 0.3^2 + 0.5^2 + 0.7^2 + 0.9^2\right] = 1.65/5 = 0.33\,,$$

which agrees up to the second decimal place with the actual value $1/3$.

1.1.4. The Distance Problem. Here we show how the concept of definite integral can be applied to more general problems. In particular we study the problem of finding the distance traveled by an object with variable velocity during a certain period of time.

If the velocity v were constant we could just multiply it by the time t: distance $= v \times t$. Otherwise we can approximate the total distance traveled by dividing the total time interval into small intervals so that in each of them the velocity varies very little and can be considered approximately constant. So, assume that the body starts moving at time t_{start} and finishes at time t_{end}, and the velocity is variable, i.e., is a function of time $v = f(t)$. We divide the time interval into n small intervals $[t_{i-1}, t_i]$ of length $\Delta t = (t_{\text{end}} - t_{\text{start}})/n$, choose some instant t_i^* between t_{i-1} and t_i, and take $v = f(t_i^*)$ as the approximate velocity of the body between t_{i-1} and t_i. Then the distance traveled during that time interval is approximately $f(t_i^*)\,\Delta t$, and the total distance can be approximated as the sum

$$\sum_{i=1}^{n} f(t_i^*)\,\Delta t$$

The result will be more accurate the larger the number of subintervals is, and the exact distance traveled will be limit of the above expression as n goes to infinity:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\,\Delta t$$

That limit turns out to be the following definite integral:

$$\int_{t_{\text{start}}}^{t_{\text{end}}} f(t)\,dt$$

1.1.5. Properties of the Definite Integral.

(1) Integral of a constant: $\displaystyle\int_a^b c\,dx = c\,(b-a)$.

(2) Linearity:

(a) $\displaystyle\int_a^b [f(x) + g(x)]\;dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(b) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

(3) Interval Additivity

(a) $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

(b) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$

(c) $\displaystyle\int_a^a f(x)\,dx = 0.$

(4) Comparison:

(a) $f(x) \geq 0 \;\Rightarrow\; \displaystyle\int_a^b f(x)\,dx \geq 0.$

(b) $f(x) \geq g(x) \;\Rightarrow\; \displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

(c) $m \leq f(x) \leq M \;\Rightarrow\; m\,(b-a) \leq \displaystyle\int_a^b f(x)\,dx \leq M\,(b-a).$

1.2. The Evaluation Theorem

1.2.1. The Evaluation Theorem. If f is a continuous function and F is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Example: Find $\int_0^1 x^2\,dx$ using the evaluation theorem.

Answer: An antiderivative of x^2 is $x^3/3$, hence:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}\,.$$

1.2.2. Indefinite Integrals. If F is an antiderivative of a function f, i.e., $F'(x) = f(x)$, then for any constant C, $F(x) + C$ is another antiderivative of $f(x)$. The family of all antiderivatives of f is called *indefinite integral* of f and represented:

$$\int f(x)\,dx = F(x) + C\,.$$

Example: $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C.$

1.2.3. Table of Indefinite Integrals. We can make an integral table just by reversing a table of derivatives.

(1) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1).$

(2) $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C.$

(3) $\displaystyle\int e^x\,dx = e^x + C.$

(4) $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C.$

(5) $\displaystyle\int \sin x\,dx = -\cos x + C.$

(6) $\displaystyle\int \cos x\,dx = \sin x + C.$

(7) $\displaystyle\int \sec^2 x\,dx = \tan x + C.$

(8) $\int \csc^2 x\, dx = -\cot x + C$.

(9) $\int \sec x \tan x\, dx = \sec x + C$.

(10) $\int \csc x \cot x\, dx = -\csc x + C$.

(11) $\int \frac{dx}{x^2+1} = \tan^{-1} x + C$.

(12) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$.

(13) $\int \frac{dx}{x\sqrt{x^2-1}}\, dx = \sec^{-1} |x| + C$.

1.2.4. Total Change Theorem. The integral of a rate of change is the total change:

$$\int_a^b F'(x)\, dx = F(b) - F(a)\,.$$

This is just a restatement of the evaluation theorem.

As an example of application we find the *net distance* or *displacement*, and the *total distance* traveled by an object that moves along a straight line with position function $s(t)$. The velocity of the object is $v(t) = s'(t)$. The net distance or displacement is the difference between the final and the initial positions of the object, and can be found with the following integral

$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)\,.$$

In the computation of the displacement the distance traveled by the object when it moves to the left (while $v(t) \leq 0$) is subtracted from the distance traveled to the right (while $v(t) \geq 0$). If we want to find the total distance traveled we need to add all distances with a positive sign, and this is accomplished by integrating the absolute value of the velocity:

$$\int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}\,.$$

Example: Find the displacement and the total distance traveled by an object that moves with velocity $v(t) = t^2 - t - 6$ from $t = 1$ to $t = 4$.

Answer: The displacement is

$$\begin{aligned}
\int_1^4 (t^2 - t - 6)\,dx &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 \\
&= \left(\frac{4^3}{3} - \frac{4^2}{2} - 6\cdot 4\right) - \left(\frac{1^3}{3} - \frac{1^2}{2} - 6\right) \\
&= -\frac{32}{3} - \left(-\frac{37}{6}\right) = \boxed{-\frac{9}{2}}
\end{aligned}$$

In order to find the total distance traveled we need to separate the intervals in which the velocity takes values of different signs. Those intervals are separated by points at which $v(t) = 0$, i.e., $t^2 - t - 6 = 0 \Rightarrow t = -2$ and $t = 3$. Since we are interested only in what happens in $[1, 4]$ we only need to look at the intervals $[1, 3]$ and $[3, 4]$. Since $v(1) = -6$, the velocity is negative in $[1, 3]$, and since $v(4) = 6$, the velocity is positive in $[3, 4]$. Hence:

$$\begin{aligned}
\int_1^4 |v(t)|\,dt &= \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt \\
&= \int_1^3 -(t^2 - t - 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt \\
&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\
&= \frac{22}{3} + \frac{17}{6} = \boxed{\frac{61}{6}}.
\end{aligned}$$

1.3. The Fundamental Theorem of Calculus

1.3.1. The Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus (FTC) connects the two branches of calculus: differential calculus and integral calculus. It says the following:

Suppose f is continuous on $[a, b]$. Then:

(1) The function
$$g(x) = \int_a^x f(t)\,dt$$
is an antiderivative of f, i.e., $g'(x) = f(x)$.

(2) (Evaluation Theorem) If F is an antiderivative of f, i.e. $F'(x) = f(x)$, then
$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

The two parts of the theorem can be rewritten like this:

(1) $\displaystyle \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

(2) $\displaystyle \int_a^b F'(x)\,dx = F(b) - F(a)$.

So the theorem states that integration and differentiation are inverse operations, i.e., the derivative of an integral of a function yields the original function, and the integral of a derivative also yields the function originally differentiated (up to a constant).

Example: Find $\displaystyle \frac{d}{dx}\int_0^{x^2} t^3\,dt$.

Answer: We solve this problem in two ways. First directly:
$$g(x) = \int_0^{x^2} t^3\,dt = \left[\frac{t^4}{4}\right]_0^{x^2} = \frac{(x^2)^4}{4} = \frac{x^8}{4}\,,$$
hence
$$g'(x) = \frac{8x^7}{4} = \boxed{2x^7}\,.$$

Second, using the FTC:

$$h(u) = \int_0^u t^3 \, dt \;\Rightarrow\; h'(u) = u^3 \,.$$

Now we have $g(x) = h(x^2)$, hence (using the chain rule):

$$g'(x) = h'(x^2) \cdot 2x = (x^2)^3 \cdot 2x = \boxed{2x^7} \,.$$

1.4. The Substitution Rule

1.4.1. The Substitution Rule. The substitution rule is a trick for evaluating integrals. It is based on the following identity between differentials (where u is a function of x):

$$du = u'\,dx\,.$$

Hence we can write:

$$\boxed{\int f(u)\,u'\,dx = \int f(u)\,du}$$

or using a slightly different notation:

$$\boxed{\int f(g(x))\,g'(x)\,dx = \int f(u)\,du}$$

where $u = g(x)$.

Example: Find $\displaystyle\int \sqrt{1+x^2}\,2x\,dx$.

Answer: Using the substitution $u = 1 + x^2$ we get

$$\begin{aligned}\int \sqrt{1+x^2}\,2x\,dx &= \int \sqrt{u}\,u'\,dx \\ &= \int \sqrt{u}\,du = \tfrac{2}{3}u^{3/2} + C \\ &= \boxed{\tfrac{2}{3}(1+x^2)^{3/2} + C}\,.\end{aligned}$$

Most of the time the only problem in using this method of integration is finding the right substitution.

Example: Find $\displaystyle\int \cos 2x\,dx$.

Answer: We want to write the integral as $\int \cos u\,du$, so $\cos u = \cos 2x \Rightarrow u = 2x$, $u' = 2$. Since we do not see any factor 2 inside the integral we write it, taking care of dividing by 2 outside the integral:

$$\begin{aligned}\int \cos 2x\,dx &= \frac{1}{2}\int \cos 2x\,2\,dx \\ &= \frac{1}{2}\int \cos u\,u'\,dx \\ &= \frac{1}{2}\int \cos u\,du \\ &= \frac{1}{2}\sin u + C\end{aligned}$$

(always remember to undo the substitution)

$$= \boxed{\frac{1}{2}\sin 2x + C}.$$

In general we need to look at the integrand as a function of some expression (which we will later identify with u) multiplied by the derivative of that expression.

Example: Find $\displaystyle\int e^{-x^2}\,x\,dx$.

Answer: We see that x is "almost", the derivative of $-x^2$, so we use the substitution $u = -x^2$, $u' = -2x$, hence in order to get u' inside the integral we do the following:

$$\begin{aligned}\int e^{-x^2}\,x\,dx &= -\frac{1}{2}\int \underbrace{e^{-x^2}}_{e^u}\,\underbrace{(-2x)\,dx}_{du} \\ &= -\frac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u + C = \boxed{-\tfrac{1}{2}e^{-x^2} + C}.\end{aligned}$$

Sometimes the substitution is hard to see until we make some ingenious transformation in the integrand.

Example: Find $\displaystyle\int \tan x\,dx$.

Answer: Here the idea is to write $\tan x = \dfrac{\sin x}{\cos x}$ and use that $(\cos x)' = -\sin x$, so we make the substitution $u = \cos x$, $u' = -\sin x$:

$$\begin{aligned}\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{u'}{u}\,dx = -\int \frac{1}{u}\,du \\ = -\ln|u| + C = \boxed{-\ln|\cos x| + C}.\end{aligned}$$

In general we need to identify inside the integral some expression of the form $f(u)\,u'$, where f is some function with a known antiderivative.

Example: Find $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$.

Answer: Let's write

$$\int \frac{e^x}{e^{2x}+1}\,dx = k\int f(u)\,u'\,dx$$

(where k is some constant to be determined later) and try to identify the function f, the argument u and its derivative u'. Since $(e^x)' = e^x$ it seems natural to chose $u = e^x$, $u' = e^x$, so $e^{2x} = u^2$ and

$$\begin{aligned}\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{u'}{u^2+1}\,dx = \int \frac{1}{u^2+1}\,du \\ = \tan^{-1} u + C = \boxed{\tan^{-1}(e^x) + C}.\end{aligned}$$

There is no much more that can be said in general, the way to learn more is just to practice.

1.4.2. Other Changes of Variable. Sometimes rather than making a substitution of the form u = function of x, we may try a change of variable of the form x = function of some other variable such as t, and write $dx = x'(t)\,dt$, where x' = derivative of x respect to t.

Example: Find $\displaystyle\int \sqrt{1-x^2}\,dx$.

Answer: Here we write $x = \sin t$, so $dx = \cos t\,dt$, $1 - x^2 = 1 - \sin^2 t = \cos^2 t$, and

$$\int \sqrt{1-x^2}\,dx = \int \underbrace{\cos t}_{x}\ \underbrace{\cos t\,dt}_{dx} = \int \cos^2 t\,dt\,.$$

Since we do not know yet how to integrate $\cos^2 t$ we leave it like this and will be back to it later (after we study integrals of trigonometric functions).

1.4.3. The Substitution Rule for Definite Integrals. When computing a definite integral using the substitution rule there are two possibilities:

(1) Compute the indefinite integral first, then use the evaluation theorem:

$$\int f(u)\,u'\,dx = F(x)\,;$$
$$\int_a^b f(u)\,u'\,dx = F(b) - F(a)\,.$$

(2) Use the substitution rule for definite integrals:

$$\int_a^b f(u)\,u'\,dx = \int_{u(a)}^{u(b)} f(u)\,du\,.$$

The advantage of the second method is that we do not need to undo the substitution.

Example: Find $\int_0^4 \sqrt{2x+1}\,dx$.

Answer: Using the first method first we compute the indefinite integral:

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \frac{1}{2}\int \sqrt{2x+1}\,2\,dx \quad (u = 2x+1)\\ &= \frac{1}{2}\int \sqrt{u}\,du\\ &= \frac{1}{3}u^{3/2} + C\\ &= \frac{1}{3}(2x+1)^{3/2} + C\,.\end{aligned}$$

Then we use it for computing the definite integral:

$$\int_0^4 \sqrt{2x+1}\,dx = \left[\frac{1}{3}(2x+1)^{3/2}\right]_0^4 = \frac{1}{3}9^{3/2} - \frac{1}{3}1^{3/2} = \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}\,.$$

In the second method we compute the definite integral directly adjusting the limits of integration after the substitution:

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_0^4 \sqrt{2x+1}\,2\,dx \qquad (u = 2x+1;\ u' = 2)$$

$$= \frac{1}{2}\int_1^9 \sqrt{u}\,du$$

(note the change in the limits of integration to $u(0) = 1$ and $u(4) = 9$)

$$= \left[\frac{1}{3}u^{3/2}\right]_1^9$$

$$= \frac{1}{3}9^{3/2} - \frac{1}{3}1^{3/2}$$

$$= \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}.$$

1.5. Integration by Parts

The method of integration by parts is based on the product rule for differentiation:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)\,,$$

which we can write like this:

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)\,.$$

Integrating we get:

$$\int f(x)\,g'(x)\,dx = \int [f(x)g(x)]'\,dx - \int g(x)f'(x)\,dx\,,$$

i.e.:

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}\,.$$

Writing $u = f(x)$, $v = g(x)$, we have $du = f'(x)\,dx$, $dv = g'(x)\,dx$, hence:

$$\boxed{\int u\,dv = uv - \int v\,du}\,.$$

Example: Integrate $\int xe^x\,dx$ by parts.

Answer: In integration by parts the key thing is to choose u and dv correctly. In this case the "right" choice is $u = x$, $dv = e^x\,dx$, so $du = dx$, $v = e^x$. We see that the choice is right because the new integral that we obtain after applying the formula of integration by parts is simpler than the original one:

$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du} = \boxed{xe^x - e^x + C}\,.$$

Usually it is a good idea to check the answer by differentiating it:

$$(xe^x - e^x + C)' = e^x + xe^x - e^x = xe^x\,.$$

A couple of additional typical examples:

Example: $\displaystyle\int x\sin x\,dx = \cdots$

$u = x$, $dv = \sin x\,dx$, so $du = dx$, $v = -\cos x$:

$$\cdots = \int \underbrace{x}_{u} \underbrace{\sin x\,dx}_{dv} = \underbrace{x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{dx}_{du}$$

$$= \boxed{-x\cos x + \sin x + C}.$$

Example: $\int \ln x\,dx = \cdots$

$u = \ln x$, $dv = dx$, so $du = \frac{1}{x}dx$, $v = x$:

$$\cdots = \int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = \underbrace{\ln x}_{u} \underbrace{x}_{v} - \int \underbrace{x}_{v} \underbrace{\frac{1}{x}\,dx}_{du}$$

$$= x\,\ln x - \int dx$$

$$= \boxed{x\,\ln x - x + C}.$$

Sometimes we need to use the formula more than once.

Example: $\int x^2 e^x\,dx = \dots$

$u = x^2$, $dv = e^x\,dx$, so $du = 2x\,dx$, $v = e^x$:

$$\cdots = \int \underbrace{x^2}_{u} \underbrace{e^x\,dx}_{dv} = x^2 e^x - \int e^x 2x\,dx = \dots$$

$u = 2x$, $dv = e^x\,dx$, so $du = 2dx$, $v = e^x$:

$$\cdots = x^2 e^x - \int \underbrace{2x}_{u} \underbrace{e^x\,dx}_{dv} = x^2 e^x - 2x e^x + \int 2e^x\,dx$$

$$= \boxed{x^2 e^x - 2x e^x + 2e^x + C}.$$

In the following example the formula of integration by parts does not yield a final answer, but an equation verified by the integral from which its value can be derived.

Example: $\int \sin x\,e^x\,dx = \dots$

$u = \sin x$, $dv = e^x\,dx$, so $du = \cos x\,dx$, $v = e^x$:

$$\cdots = \int \underbrace{\sin x}_{u}\,\underbrace{e^x\,dx}_{dv} = \sin x \cdot e^x - \int e^x \cos x\,dx = \ldots$$

$u = \cos x$, $dv = e^x\,dx$, so $du = -\sin x\,dx$, $v = e^x$:

$$\begin{aligned} \ldots &= \sin x \cdot e^x - \int \underbrace{\cos x}_{u}\,\underbrace{e^x\,dx}_{dv} \\ &= \sin x \cdot e^x - \cos x \cdot e^x - \int e^x \sin x\,dx \end{aligned}$$

Hence the integral $I = \int \sin x\, e^x\,dx$ verifies

$$I = \sin x \cdot e^x - \cos x \cdot e^x - I\,,$$

i.e.,

$$2I = \sin x \cdot e^x - \cos x \cdot e^x\,,$$

hence

$$\boxed{I = \tfrac{1}{2}e^x(\sin x - \cos x) + C}\,.$$

1.5.1. Integration by parts for Definite Integrals. Combining the formula of integration by parts with the Evaluation Theorem we get:

$$\boxed{\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx}\,.$$

Example: $\displaystyle\int_0^1 \tan^{-1} x\,dx = \cdots$

$u = \tan^{-1} x$, $dv = dx$, so $du = \dfrac{1}{1+x^2}\,dx$, $v = x$:

$$\begin{aligned} \cdots = \int_0^1 \underbrace{\tan^{-1} x}_{u}\ \underbrace{dx}_{dv} &= [\,\underbrace{\tan^{-1} x}_{u} \cdot \underbrace{x}_{v}\,]_0^1 - \int_0^1 \underbrace{x}_{v}\,\underbrace{\frac{1}{1+x^2}\,dx}_{du} \\ &= \left[\tan^{-1} 1 \cdot 1 - \tan^{-1} 0 \cdot 0\right] - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx \end{aligned}$$

The last integral can be computed with the substitution $t = 1 + x^2$, $dt = 2x\,dx$:

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{t}\,dt = \frac{1}{2}\,[\ln t]_1^2 = \frac{\ln 2}{2}\,.$$

Hence the original integral is:

$$\int_0^1 \tan^{-1} x\,dx = \boxed{\frac{\pi}{4} - \frac{\ln 2}{2}}\,.$$

1.5.2. Reduction Formulas. Assume that we want to find the following integral for a given value of $n > 0$:

$$\int x^n e^x\,dx\,.$$

Using integration by parts with $u = x^n$ and $dv = e^x\,dx$, so $v = e^x$ and $du = nx^{n-1}\,dx$, we get:

$$\boxed{\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx}\,.$$

On the right hand side we get an integral similar to the original one but with x raised to $n-1$ instead of n. This kind of expression is called a *reduction formula*. Using this same formula several times, and taking into account that for $n = 0$ the integral becomes $\int e^x\,dx = e^x + C$, we can evaluate the original integral for any n. For instance:

$$\begin{aligned}
\int x^3 e^x\,dx &= x^3 e^x - 3\int x^2 e^x\,dx \\
&= x^3 e^x - 3(x^2 e^x - 2\int x e^x\,dx) \\
&= x^3 e^x - 3(x^2 e^x - 2(x e^x - \int e^x\,dx)) \\
&= x^3 e^x - 3(x^2 e^x - 2(x e^x - e^x)) + C \\
&= \boxed{x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C}\,.
\end{aligned}$$

Another example:

$$\begin{aligned}\int \sin^n x\,dx &= \int \underbrace{\sin^{n-1} x}_{u}\,\underbrace{\sin x\,dx}_{dv} \\ &= -\sin^{n-1} x \cos x + (n-1) \int \underbrace{\cos^2 x}_{1-\sin^2 x} \sin^{n-2}\;dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2}\;dx \\ &\qquad - (n-1) \int \sin^n x\,dx\end{aligned}$$

Adding the last term to both sides and dividing by n we get the following reduction formula:

$$\boxed{\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\,dx}.$$

1.6. Trigonometric Integrals and Trigonometric Substitutions

1.6.1. Trigonometric Integrals. Here we discuss integrals of powers of trigonometric functions. To that end the following *half-angle identities* will be useful:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)\,,$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)\,.$$

Remember also the identities:

$$\sin^2 x + \cos^2 x = 1\,,$$
$$\sec^2 x = 1 + \tan^2 x\,.$$

1.6.1.1. *Integrals of Products of Sines and Cosines.* We will study now integrals of the form

$$\int \sin^m x \cos^n x\,dx\,,$$

including cases in which $m = 0$ or $n = 0$, i.e.:

$$\int \cos^n x\,dx\,; \qquad \int \sin^m x\,dx\,.$$

The simplest case is when either $n = 1$ or $m = 1$, in which case the substitution $u = \sin x$ or $u = \cos x$ respectively will work.

Example: $\displaystyle\int \sin^4 x \cos x\,dx = \cdots$

$(u = \sin x,\ du = \cos x\,dx)$

$$\cdots = \int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{\sin^5 x}{5} + C}\,.$$

More generally if at least one exponent is odd then we can use the identity $\sin^2 x + \cos^2 x = 1$ to transform the integrand into an expression containing only one sine or one cosine.

Example:

$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^2 x \cos x \, dx \\ &= \int \sin^2 x \,(1 - \sin^2 x) \cos x \, dx = \cdots\end{aligned}$$

$(u = \sin x,\ du = \cos x \, dx)$

$$\begin{aligned}\cdots = \int u^2 \,(1 - u^2) \, du &= \int (u^2 - u^4) \, du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}.\end{aligned}$$

If all the exponents are even then we use the half-angle identities.

Example:

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \tfrac{1}{2}(1 - \cos 2x)\tfrac{1}{2}(1 + \cos 2x) \, dx \\ &= \frac{1}{4} \int (1 - \cos^2 2x) \, dx \\ &= \frac{1}{4} \int (1 - \tfrac{1}{2}(1 + \cos 4x)) \, dx \\ &= \frac{1}{8} \int (1 - \cos 4x) \, dx \\ &= \frac{x}{8} - \frac{\sin 4x}{32} + C \,.\end{aligned}$$

1.6.1.2. *Integrals of Secants and Tangents.* The integral of $\tan x$ can be computed in the following way:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = \boxed{-\ln|u| + C = -\ln|\cos x| + C},$$

where $u = \cos x$. Analogously

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sin x| + C},$$

where $u = \sin x$.

The integral of $\sec x$ is a little tricky:

$$\int \sec x\,dx = \int \frac{\sec x\,(\tan x + \sec x)}{\sec x + \tan x}\,dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx =$$
$$\int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C},$$

where $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\,dx$.

Analogously:

$$\int \csc x\,dx = \boxed{-\ln|\csc x + \cot x| + C}.$$

More generally an integral of the form

$$\int \tan^m x \sec^n x\,dx$$

can be computed in the following way:

(1) If m is odd, use $u = \sec x$, $du = \sec x \tan x\,dx$.
(2) If n is even, use $u = \tan x$, $du = \sec^2 x\,dx$.

Example: $\displaystyle\int \tan^3 x \sec^2 x\,dx = \cdots$

Since in this case m is odd and n is even it does not matter which method we use, so let's use the first one:

$(u = \sec x,\ du = \sec x \tan x\,dx)$

$$\begin{aligned}\cdots = \int \underbrace{\tan^2 x}_{u^2-1}\ \underbrace{\sec x}_{u}\ \underbrace{\tan x \sec x\,dx}_{du} &= \int (u^2-1)u\,du \\ &= \int (u^3 - u)\,du \\ &= \frac{u^4}{4} - \frac{u^2}{2} + C \\ &= \boxed{\tfrac{1}{4}\sec^4 x - \tfrac{1}{2}\sec^2 x + C}.\end{aligned}$$

Next let's solve the same problem using the second method:

$(u = \tan x,\ du = \sec^2 x\, dx)$

$$\int \underbrace{\tan^3 x}_{u^3}\ \underbrace{\sec^2 x\, dx}_{du} = \int u^3\, du = \frac{u^4}{4} + C = \boxed{\tfrac{1}{4}\tan^4 x + C}\,.$$

Although this answer looks different from the one obtained using the first method it is in fact equivalent to it because they differ in a constant:

$$\tfrac{1}{4}\tan^4 x = \tfrac{1}{4}(\sec^2 x - 1)^2 = \underbrace{\tfrac{1}{4}\sec^4 x - \tfrac{1}{2}\sec^2 x}_{\text{previous answer}} + \tfrac{1}{4}\,.$$

1.6.2. Trigonometric Substitutions. Here we study substitutions of the form $x =$ some trigonometric function.

Example: Find $\displaystyle\int \sqrt{1-x^2}\, dx$.

Answer: We make $x = \sin t$, $dx = \cos t\, dt$, hence

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t\,,$$

and

$$\begin{aligned}
\int \sqrt{1-x^2}\, dx &= \int \cos t\ \cos t\, dt \\
&= \int \cos^2 t\, dt \\
&= \int \tfrac{1}{2}(1+\cos 2t)\, dt && \text{(half-angle identity)} \\
&= \frac{t}{2} + \frac{\sin 2t}{4} + C \\
&= \frac{t}{2} + \frac{2\sin t\cos t}{4} + C && \text{(double-angle identity)} \\
&= \frac{t}{2} + \frac{\sin t\sqrt{1-\sin^2 t}}{2} + C \\
&= \boxed{\frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2} + C}\,.
\end{aligned}$$

The following substitutions are useful in integrals containing the following expressions:

expression	substitution	identity
$a^2 - u^2$	$u = a \sin t$	$1 - \sin^2 t = \cos^2 t$
$a^2 + u^2$	$u = a \tan t$	$1 + \tan^2 t = \sec^2 t$
$u^2 - a^2$	$u = a \sec t$	$\sec^2 t - 1 = \tan^2 t$

So for instance, if an integral contains the expression $a^2 - u^2$, we may try the substitution $u = a \sin t$ and use the identity $1 - \sin^2 t = \cos^2 t$ in order to transform the original expression in this way:

$$a^2 - u^2 = a^2(1 - \sin^2 t) = a^2 \cos^2 t\,.$$

Example:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{9 - x^2}}\,dx &= 27 \int \frac{\sin^3 t \cos t}{\sqrt{1 - \sin^2 t}}\,dt && (x = 3 \sin t) \\
&= 27 \int \sin^3 t\,dx \\
&= 27 \int (1 - \cos^2 t)\,\sin t\,dx \\
&= 27 \left(- \cos t + \frac{\cos^3 t}{3} \right) + C \\
&= 27 \left(- \sqrt{1 - \sin^2 t} + \tfrac{1}{3}(1 - \sin^2 t)^{3/2} \right) + C \\
&= \boxed{-9\sqrt{9 - x^2} + \tfrac{1}{3}(9 - x^2)^{3/2} + C}\,.
\end{aligned}$$

where $x = 3 \sin t$, $dx = 3 \cos t\,dt$.

Example:

$$
\begin{aligned}
\int \sqrt{9+4x^2}\,dx &= 2\int \sqrt{\tfrac{9}{4}+x^2}\,dx \qquad\qquad (x=\tfrac{3}{2}\tan t)\\
&= 2\int \frac{3}{2}\sqrt{1+\tan^2 t}\,\frac{3}{2}\sec^2 t\,dt\\
&= \frac{9}{2}\int \sec^3 t\,dt\\
&= \frac{9}{4}(\sec t\tan t + \ln|\sec t+\tan t|) + C_1\\
&= \frac{9}{4}\left(\tfrac{2}{3}x\sqrt{1+\tfrac{4}{9}x^2} + \ln\left|\tfrac{2}{3}x + \sqrt{1+\tfrac{4}{9}x^2}\right|\right) + C_1\\
&= \boxed{\frac{x\sqrt{9+4x^2}}{2} + \frac{9}{4}\ln|2x+\sqrt{9+4x^2}| + C}\,.
\end{aligned}
$$

where $x = \frac{3}{2}\tan t$, $dx = \frac{3}{2}\sec^2 t\,dt$

Example:

$$
\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2 t-1}}{\sec t}\,\sec t\tan t\,dt \qquad\qquad (x=\sec t)\\
&= \int \tan^2 t\,dt\\
&= \tan t - t + C\\
&= \sqrt{\sec^2 t-1} - t + C\\
&= \sqrt{x^2-1} - \sec^{-1} x + C\,.
\end{aligned}
$$

where $x = \sec t$, $dx = \sec t\tan t\,dt$.

1.7. Partial Fractions

1.7.1. Rational Functions and Partial Fractions. A rational function is a quotient of two polynomials:

$$R(x) = \frac{P(x)}{Q(x)}\,.$$

Here we discuss how to integrate rational functions. The idea consists of rewriting the rational function as a sum of simpler fractions called *partial fractions*. This can be done in the following way:

(1) Use long division of polynomials to get a quotient $p(x)$ and a remainder $r(x)$. Then write:

$$R(x) = \frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}\,,$$

where the degree of $r(x)$ is less than that of $Q(x)$.

(2) Factor the denominator $Q(x) = q_1(x)q_2(x)\dots q_n(x)$, where each factor $q_i(x)$ is either linear $ax+b$, or irreducible quadratic ax^2+bx+c, or a power of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$.

(3) Decompose $r(x)/Q(x)$ into partial fractions of the form:

$$\frac{r(x)}{Q(x)} = F_1(x) + F_2(x) + F_3(x) + \cdots$$

where each fraction is of the form

$$F_i(x) = \frac{A}{(ax+b)^k}$$

or

$$F_i(x) = \frac{Ax+B}{(ax^2+bx+c)^k}\,,$$

where $1 \le k \le n$ (n is the exponent of $ax+b$ or ax^2+bx+c in the factorization of $Q(x)$.)

Example: Decompose the following rational function into partial fractions:

$$R(x) = \frac{x^3+x^2+2}{x^2-1}$$

Answer:

(1) $\dfrac{x^3+x^2+2}{x^2-1} = x+1+\dfrac{x+3}{x^2-1}$

(2) $x^2-1=(x+1)(x-1)$.

(3) $\dfrac{x+3}{(x+1)(x-1)} = \dfrac{A}{x+1}+\dfrac{B}{x-1}$.

Multiplying by $(x+1)(x-1)$ we get:

$$x+3 = A(x-1)+B(x+1)\,. \qquad (*)$$

Now there are two ways of finding A and B:

Method 1. Expand the right hand side of (*), collect terms with the same power of x, and identify coefficients of the polynomials obtained on both sides:

$$x+3 = (A+B)\,x + (B-A)\,,$$

Hence:

$$\begin{cases} 1 = A+B & \text{(coefficient of } x\text{)} \\ 3 = -A+B & \text{(constant term)} \end{cases}$$

Method 2. In (*) give x two different values (as many as the number of coefficients to determine), say $x=1$ and $x=-1$. We get:

$$\begin{cases} 4 = 2B & (x=1) \\ 2 = -2A & (x=-1) \end{cases}$$

The solution to the system of equations obtained in either case is $A=-1$, $B=2$, so:

$$\frac{x+3}{(x+1)(x-1)} = -\frac{1}{x+1}+\frac{2}{x-1}\,.$$

Finally:

$$R(x) = \frac{x^3+x^2+2}{x^2-1} = x+1-\frac{1}{x+1}+\frac{2}{x-1}\,.$$

1.7.2. Factoring a Polynomial. In order to factor a polynomial $Q(x)$ (with real coefficients) into linear of irreducible quadratic factors, first solve the algebraic equation:

$$Q(x) = 0\,.$$

Then for each real root r write a factor of the form $(x-r)^k$ where k is the multiplicity of the root. For each pair of conjugate complex roots $r, \bar{r}$ write a factor $(x^2-sx+p)^k$, where $s=r+\bar{r}$, $p=r\cdot\bar{r}$, and k is the multiplicity of those roots. Finally multiply by the leading coefficient of $Q(x)$.

Note that the equation $Q(x) = 0$ is sometimes hard to solve, or only the real roots can be easily found (when they are integral or rational they can be found by Ruffini's rule, or just by trial and error). In that case we get as many roots as we can, and divide $Q(x)$ by the factors found. The quotient is another polynomial $q(x)$ which we must now try to factor. So pose the algebraic equation

$$q(x) = 0$$

and try to solve it for this new (and simpler) polynomial.

Example: Factor the polynomial

$$Q(x) = x^6 - x^5 - 15x^4 + 5x^3 + 70x^2 + 12x - 72\,.$$

Answer: The roots of $Q(x)$ are 1 (simple), -2 (triple) and 3 (double), hence:

$$Q(x) = (x - 1)(x + 2)^3(x - 3)^2\,.$$

Example: Factor $Q(x) = x^3 + 2x^2 + 2x + 1$.

Answer: $Q(x)$ has a simple real root $x = -1$. After dividing $Q(x)$ by $x + 1$ we get the polynomial $x^2 + x + 1$, which is irreducible (it has only complex roots), so we factor $Q(x)$ like this:

$$Q(x) = (x + 1)(x^2 + x + 1)\,.$$

1.7.3. Decomposing Into Partial Fractions. Assume that $Q(x)$ has already been factored and degree of $r(x)$ is less than degree of $Q(x)$. Then $r(x)/Q(x)$ is decomposed into partial fractions in the following way:

(1) For each factor of the form $(x - r)^k$ write

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_k}{(x-r)^k}\,,$$

where $A_1 \dots A_k$ are coefficients to be determined.

(2) For each factor of the form $(ax^2 + bx + c)^k$ write

$$\frac{B_1x + C_1}{ax^2+bx+c} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx + C_k}{(ax^2+bx+c)^k}$$

where $B_1 \dots B_k$ and $C_1 \dots C_k$ are coefficients to be determined.

(3) Multiply by $Q(x)$ and simplify. This leads to an expression of the form

$$r(x) = \text{some polynomial containing the} \quad \text{indeterminate coefficients } A_i, B_i, C_i\,.$$

Finally determine the coefficients A_i, B_i, C_i. One way of doing this is by identifying coefficients of the polynomials on both sides of the last expression. Another way is to write a system of equations with unknowns A_i, B_i, C_i by giving x various values.

Example: Decompose the following rational function into partial fractions:

$$R(x) = \frac{4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3}{(x-1)^2\,(x^2+x+1)^2}\,.$$

Answer: The denominator is already factored, so we proceed with the next step:

$$\frac{4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3}{(x-1)^2\,(x^2+x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}\,.$$

Next we multiply by the denominator:

$$\begin{aligned} 4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3 &= \\ A(x-1)(x^2+x+1)^2 &+ B(x^2+x+1)^2 \\ + (Cx+D)(x-1)^2(x^2+x+1) &+ (Ex+F)(x-1)^2 = \\ (A+C)x^5 &+ (A-C+D+B)x^4 \\ + (A+2B+E-D)x^3 &+ (-A+3B-C-2E+F)x^2 \\ + (-A+2B+C-D+E-2F)x &+ (-A+B+D+F)\,. \end{aligned}$$

Identifying coefficients on both sides we get:

$$\left\{ \begin{array}{rcrcrcrcrcrcr} A & & & + & C & & & & & & & = & 4 \\ A & + & B & - & C & + & D & & & & & = & -2 \\ A & + & 2B & & & - & D & + & E & & & = & 2 \\ -A & + & 3B & - & C & & & - & 2E & + & F & = & -8 \\ -A & + & 2B & + & C & - & D & + & E & - & 2F & = & -2 \\ -A & + & B & & & + & D & & & + & F & = & -3 \end{array} \right.$$

The solution to this system of equations is $A = 2, B = -1, C = 2, D = -1, E = 1, F = 1$, hence:

$$\frac{4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3}{(x-1)^2\,(x^2+x+1)^2} = \\ \frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{2x-1}{x^2+x+1} + \frac{x+1}{(x^2+x+1)^2}$$

1.7.4. Integration of Rational Functions. After decomposing the rational function into partial fractions all we need to do is to integrate expressions of the form $A/(x-r)^k$ and $(Bx+C)/(ax^2+bx+c)^k$. For the former we get:

$$\int \frac{A}{(x-r)^k}\,dx = -\frac{A}{(k-1)(x-r)^{k-1}} + C \qquad \text{if } k \neq 1$$

$$\int \frac{A}{x-r}\,dx = A \ln|x-r| + C \qquad \text{if } k = 1$$

The latter are more involved, but the following are particularly simple special cases:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C$$

$$\int \frac{x}{(x^2+a^2)^k}\,dx = -\frac{1}{2(k-1)(x^2+a^2)^{k-1}} + C \quad (k \neq 1)$$

Example: Find the following integral: $\displaystyle\int \frac{x^3 - x^2 - 7x + 8}{x^2 - 4x + 4}\,dx\,.$

Answer: First we decompose the integrand into partial fractions:

(1) $\displaystyle\frac{x^3 - x^2 - 7x + 8}{x^2 - 4x + 4} = x + 3 + \frac{x-4}{x^2 - 4x + 4}$

(2) $x^2 - 4x + 4 = (x-2)^2$.

(3) $\displaystyle\frac{x-4}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$x - 4 = A(x-2) + B$

$$x = 2 \quad \Rightarrow \quad -2 = B$$
$$x = 3 \quad \Rightarrow \quad -1 = A + B$$

So $A = 1$, $B = -2$, and

$$\frac{x-4}{x^2-4x+4} = \frac{1}{x-2} - \frac{2}{(x-2)^2}$$

Hence:

$$\frac{x^3-x^2-7x+8}{x^2-4x+4} = x+3+\frac{1}{x-2} - \frac{2}{(x-2)^2}\,.$$

Finally we integrate:

$$\begin{aligned}\int \frac{x^3-x^2-7x+8}{(x-2)^2}\,dx &= \int (x+3)\,dx + \int \frac{1}{x-2}\,dx - \int \frac{2}{(x-2)^2}\,dx \\ &= \boxed{\frac{x^2}{2} + 3x + \ln|x-2| + \frac{2}{x-2} + C}\,.\end{aligned}$$

1.7.5. Completing the Square. Many integrals containing an irreducible (no real roots) quadratic polynomial $ax^2 + bx + c$ can be simplified by completing the square, i.e., writing the polynomial as $u^2 + r$ where $u = px + q$, e.g.:

$$x^2 + 2x + 2 = (x+1)^2 + 1\,.$$

In general:

$$ax^2 + bx + c = \left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2-4ac}{4a}\,.$$

If $a = 1$ the formula can be simplified:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\,.$$

This result is of the form $u^2 \pm A^2$, where $u = x + b/2$.

Example:

$$\begin{aligned}\int \frac{1}{x^2+6x+10}\,dx &= \int \frac{1}{(x+3)^2+1}\,dx \\ &= \int \frac{1}{u^2+1}\,du \qquad\qquad (u = x+3) \\ &= \tan^{-1} u + C \\ &= \boxed{\tan^{-1}(x+3) + C}\,.\end{aligned}$$

Example:

$$\begin{aligned}
\int \sqrt{x^2 - 4x + 5}\, dx &= \int \sqrt{(x-2)^2 + 1}\, dx \\
&= \int \sqrt{u^2 + 1}\, du && (u = x - 2) \\
&= \int \sqrt{\tan^2 t + 1} \cdot \sec^2 t\, dt && (u = \tan t) \\
&= \int \sec^3 t\, dt \\
&= \frac{\sec t \tan t}{2} + \frac{1}{2} \ln |\sec t + \tan t| + C \\
&= \frac{u\sqrt{u^2 + 1}}{2} + \frac{1}{2} \ln |u + \sqrt{u^2 + 1}| + C \\
&= \frac{(x-2)\sqrt{x^2 - 4x + 5}}{2} \\
&\quad + \frac{1}{2} \ln \left| (x-2) + \sqrt{x^2 - 4x + 5} \right| + C\,.
\end{aligned}$$

1.8. Integration using Tables and CAS

The use of tables of integrals and Computer Algebra Systems allow us to find integrals very quickly without having to perform all the steps for their computation. However we often need to modify slightly the original integral and perhaps complete or simplify the answer.

Example: Find the following integral using the tables at the end of Steward's book:

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \cdots .$$

Answer: In the tables we find the following formula No. 41:

$$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C\,,$$

hence, letting $a = 1$, $u = x$ we get the answer:

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \boxed{\sqrt{x^2-1} - \cos^{-1}\frac{1}{|x|} + C}\,.$$

Example: Find the integral:

$$\int \frac{x^2}{\sqrt{9+4x^2}}\,dx = \cdots$$

Answer: In the tables the formula that resembles this integral most is No. 26:

$$\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C\,,$$

hence letting $a = 3$, $u = 2x$:

$$\begin{aligned}\int \frac{x^2}{\sqrt{9+4x^2}}\,dx &= \frac{1}{8}\int \frac{u^2\,du}{\sqrt{a^2+u^2}} \\ &= \frac{1}{8}\left\{\frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)\right\} + C \\ &= \boxed{\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\ln\left(2x+\sqrt{9+4x^2}\right) + C}\,.\end{aligned}$$

Example: Find the same integral using Maple.

Answer: In Maple we enter at the prompt:

```
> int(x^2/sqrt(9+4*x^2),x);
```

and it returns:

$$\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\operatorname{arcsinh}\left(\frac{2}{3}x\right)$$

First we notice that the answer omits the constant C. On the other hand, it involves an inverse hyperbolic function:

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{1+x^2}\right),$$

hence the answer provided by Maple is:

$$\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\ln\left(\frac{2x}{3} + \sqrt{1+\frac{4x^2}{9}}\right) =$$
$$\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\ln\left(2x + \sqrt{9+4x^2}\right) + \frac{9}{32}\ln(3),$$

so it differs from the answer found using the tables in a constant $\frac{9}{32}\ln(3)$ which can be absorbed into the constant of integration.

1.9. Numerical Integration

Sometimes the integral of a function cannot be expressed with *elementary functions*, i.e., polynomial, trigonometric, exponential, logarithmic, or a suitable combination of these. However, in those cases we still can find an approximate value for the integral of a function on an interval.

1.9.1. Trapezoidal Approximation. A first attempt to approximate the value of an integral $\int_a^b f(x)\,dx$ is to compute its Riemann sum:

$$R = \sum_{i=1}^{n} f(x_i^*)\,\Delta x\,.$$

Where $\Delta x = x_i - x_{i-1} = (b-a)/n$ and x_i^* is some point in the interval $[x_{i-1}, x_i]$. If we choose the left endpoints of each interval, we get the *left-endpoint approximation*:

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = (\Delta x)\{f(x_0) + f(x_1) + \cdots + f(x_{n-1})\}\,,$$

Similarly, by choosing the right endpoints of each interval we get the *right-endpoint approximation*:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = (\Delta x)\{f(x_1) + f(x_2) + \cdots + f(x_n)\}\,.$$

The *trapezoidal approximation* is the average of L_n and R_n:

$$T_n = \frac{1}{2}(L_n + R_n) = \frac{\Delta x}{2}\{f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\}\,.$$

Example: Approximate $\int_0^1 x^2\,dx$ with trapezoidal approximation using 4 intervals.

Solution: We have $\Delta x = 1/4 = 0.25$. The values for x_i and $f(x_i) = x_i^2$ can be tabulated in the following way:

i	x_i	$f(x_i)$
0	0	0
1	0.25	0.0625
2	0.5	0.25
3	0.75	0.5625
4	1	1

Hence:

$$L_4 = 0.25 \cdot (0 + 0.0625 + 0.25 + 0.5625) = 0.218750\,,$$

$$R_4 = 0.25 \cdot (0.0625 + 0.25 + 0.5625 + 1) = 0.468750\,.$$

So:

$$T_4 = \frac{1}{2}(L_4 + R_4) = \frac{1}{2}(0.218750 + 0.468750) = 0.34375\,.$$

Compare to the exact value of the integral, which is $1/3 = 0.3333\ldots$.

1.9.2. Midpoint Approximation. Alternatively, in the Riemann sum we can use the middle point $\overline{x}_i = (x_{i-1} + x_i)/2$ of each interval $[x_{i-1}, x_i]$. Then the *midpoint approximation* of $\int_a^b f(x)\,dx$ is

$$M_n = (\Delta x)\{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\}\,.$$

Example: Approximate $\int_0^1 x^2\,dx$ with midpoint approximation using 4 intervals.

Solution: We have:

i	$\overline{x}_i$	$f(\overline{x}_i)$
1	0.125	0.015625
2	0.375	0.140625
3	0.625	0.390625
4	0.875	0.765625

Hence:

$$\begin{aligned} M_4 &= 0.25 \cdot (0.015625 + 0.140625 + 0.390625 + 0.765625) \\ &= 0.328125\,. \end{aligned}$$

1.9.3. Simpson's Approximation. *Simpson's approximation* is a weighted average of the trapezoidal and midpoint approximations associated to the intervals $[x_0, x_2]$, $[x_2, x_4]$, $\ldots$, $[x_{n-2}, x_n]$ (of length

$2\Delta x$ each):

$$
\begin{aligned}
S_{2n} &= \frac{1}{3}(2M_n + T_n) \\
&= \frac{1}{3}\Big[2(2\Delta x)\{f(x_1) + f(x_3) + \cdots + f(x_{2n-1})\} \\
&\qquad + \frac{2\Delta x}{2}\{f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)\}\Big] \\
&= \frac{\Delta x}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\
&\qquad\qquad + 2f(x_{2n-2)} + 4f(x_{2n-1}) + f(x_{2n})\}\,.
\end{aligned}
$$

Example: Approximate $\int_0^1 x^2\,dx$ with Simpson's approximation using 8 intervals.

Solution: We use the previous results and get:

$$S_8 = \frac{1}{3}(2M_4 + T_4) = \frac{1}{3}(2 \cdot 0.328125 + 0.34375) = 1/3\,.$$

Note: in this particular case Simpson's approximation gives the exact value—in general it just gives a good approximation.

1.9.4. Error Bounds. Here we give a way to estimate the error or difference E between the actual value of an integral and the value obtained using a numerical approximation.

1.9.4.1. *Error Bound for the Trapezoidal Approximation.* Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then the error E_T in the trapezoidal approximation verifies:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}\,.$$

1.9.4.2. *Error Bound for the Midpoint Approximation.* Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then the error E_M in the trapezoidal approximation verifies:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}\,.$$

1.9.4.3. *Error Bound for the Simpson's Rule.* Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. Then the error E_S in the Simpson's rule verifies:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}\,.$$

Example: Approximate the value of π using the trapezoidal, midpoint and Simpson's approximations of

$$\int_0^1 \frac{4}{1+x^2}\,dx$$

for $n = 4$. Estimate the error.

Answer: First note that:

$$4\int_0^1 \frac{1}{1+x^2}\,dx = 4\left[\tan^{-1} x\right]_0^1 = 4\frac{\pi}{4} = \pi\,,$$

so by approximating the given integral we are in fact finding approximated values for π.

Now we find the requested approximations:

(1) Trapezoidal approximation:

$$\begin{aligned} T_4 &= \frac{1/4}{2}\{f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1)\} \\ &= \boxed{3.131176470}\,. \end{aligned}$$

For estimating the error we need the second derivative of $f(x) = 4/(1+x^2)$, which is $f''(x) = 8(3x^2-1)/(1+x^2)^3$ so we have

$$\begin{aligned} |f''(x)| = \frac{8|3x^2-1|}{|1+x^2|^3} &\le \frac{8(3x^2+1)}{(1+x^2)^3} \\ &\le \frac{8(3\cdot 1^2+1)}{1} = 32 \end{aligned}$$

for $0 \le x \le 1$, hence

$$|E_T| \le \frac{32\cdot(1-0)^3}{12\cdot 4^2} = 0.1666\ldots$$

(2) Midpoint approximation:

$$\begin{aligned} M_4 &= \frac{1}{4}\{f(1/8) + f(3/8) + f(5/8) + f(7/8)\} \\ &= \boxed{3.146800518}\,. \end{aligned}$$

The error estimate is:

$$|E_M| \le \frac{32\cdot(1-0)^3}{24\cdot 4^2} = 0.08333\ldots$$

(3) Simpson's rule:

$$\begin{aligned} S_4 &= \frac{1/4}{3}\{f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)\} \\ &= \boxed{3.141568627} \end{aligned}$$

For the error estimate we now need the fourth derivative:

$$f^{(4)}(x) = 96(5x^4 - 10x^2 + 1)/(1 + x^2)^5\,,$$

so

$$|f^{(4)}(x)| \leq \frac{96(5 + 10 + 1)}{1} = 1536$$

for $0 \leq x \leq 1$. Hence the error estimate is

$$|E_S| \leq \frac{1536 \cdot (1 - 0)^5}{180 \cdot 4^4} = 0.0333\ldots$$

1.10. Improper Integrals

1.10.1. Improper Integrals. Up to now we have studied integrals of the form

$$\int_a^b f(x)\,dx\,,$$

where f is a *continuous* function defined on the *closed* and *bounded* interval $[a,b]$. *Improper integrals* are integrals in which one or both of these conditions are not met, i.e.,

(1) The interval of integration is not bounded:

$$[a,+\infty)\,,\qquad (-\infty,a]\,,\qquad (-\infty,+\infty)\,,$$

e.g.:

$$\int_1^\infty \frac{1}{x^2}\,dx\,.$$

(2) The integrand has an infinite discontinuity at some point c in $[a,b]$:

$$\lim_{x\to c} f(x) = \pm\infty\,.$$

e.g.:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx\,.$$

1.10.2. Infinite Limits of Integration. Improper Integrals of Type 1. In case one of the limits of integration is infinite, we define:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx\,,$$

or

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx\,.$$

If both limits of integration are infinite, then we choose any c and define:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,.$$

If the limits defining the integral exist the integral is called *convergent*, otherwise it is called *divergent*.

Remark: Sometimes we write $[F(x)]_a^\infty$ as an abbreviation for

$$[F(x)]_a^\infty = \lim_{t\to\infty}\,[F(x)]_a^t\,.$$

Analogously:

$$[F(x)]_{-\infty}^{a} = \lim_{t\to-\infty} [F(x)]_t^a \ ,$$

and

$$[F(x)]_{-\infty}^{\infty} = [F(x)]_{-\infty}^{c} + [F(x)]_c^{\infty} = \lim_{t\to-\infty} [F(x)]_t^c + \lim_{t\to\infty} [F(x)]_c^t \ .$$

Example:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1\,,$$

or in simplified notation:

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1\,.$$

Example: For what values of p is the following integral convergent?:

$$\int_1^\infty \frac{1}{x^p}\,dx\,.$$

Answer: If $p = 1$ then we have

$$\int_1^t \frac{1}{x}\,dx = [\ln x]_1^t = \ln t\,,$$

so

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \ln t = \infty\,,$$

and the integral is divergent. Now suppose $p \neq 1$:

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{1}{1-p}\left\{\frac{1}{t^{p-1}} - 1\right\}$$

If $p > 1$ then $p - 1 > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\left\{\frac{1}{t^{p-1}} - 1\right\} = 0\,,$$

hence the integral is convergent. On the other hand if $p < 1$ then $p - 1 < 0$, $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\left\{t^{1-p} - 1\right\} = \infty\,,$$

hence the integral is divergent. So:

$$\int_1^\infty \frac{1}{x^p}\,dx \ \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

1.10.3. Infinite Integrands. Improper Integrals of Type 2. Assume f is defined in $[a,b)$ but

$$\lim_{x\to b^-} f(x) = \pm\infty\,.$$

Then we define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx\,.$$

Analogously, if f is defined in $(a,b]$ but

$$\lim_{x\to a^+} f(x) = \pm\infty\,.$$

Then we define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx\,.$$

Finally, if $f(x)$ has an infinite discontinuity at c inside $[a,b]$, then the definition is

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

If the limits defining the integral exist the integral is called *convergent*, otherwise it is called *divergent*.

Remark: If the interval of integration is $[a,b)$ sometimes we write $[F(x)]_a^b$ as an abbreviation for $\lim_{t\to b^-}[F(x)]_a^t$—and analogously for intervals of the form $(a,b]$.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^-}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^-}\left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^-}\left(2-2\sqrt{t}\right) = 2\,,$$

or in simplified notation:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^1 = \lim_{t\to 0^-}\left(2-2\sqrt{t}\right) = 2\,.$$

Example: Evaluate $\displaystyle\int_0^1 \ln x\,dx$.

Answer: The function $\ln x$ has a vertical asymptote at $x = 0$ because $\lim_{x\to 0^+} \ln x = -\infty$. Hence:

$$\begin{aligned}
\int_0^1 \ln x\,dx &= \lim_{t\to 0^+} \int_t^1 \ln x\,dx \\
&= \lim_{t\to 0^+} [x\ln x - x]_t^1 \\
&= \lim_{t\to 0^+} \{(1\ln 1 - 1) - (t\ln t - t)\} \\
&= \lim_{t\to 0^+} \{t - 1 - t\ln t\} \qquad\qquad (\lim_{t\to 0^+} t\ln t = 0) \\
&= \boxed{-1}\,.
\end{aligned}$$

1.10.4. Comparison Test for Improper Integrals. Suppose f and g are continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq 0$.

(1) If $\int_a^\infty f(x)\,dx$ is convergent then $\int_a^\infty g(x)\,dx$ is convergent.
(2) If $\int_a^\infty g(x)\,dx$ is divergent then $\int_a^\infty f(x)\,dx$ is divergent.

A similar statement holds for type 2 integrals.

Example: Prove that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent.

Answer: We have:

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx\,.$$

The first integral on the right hand side is an ordinary definite integral so we only need to show that the second integral is convergent. In fact, for $x \geq 1$ we have $x^2 \geq x$, so $e^{-x^2} \leq e^{-x}$. On the other hand:

$$\int_1^t e^{-x}\,dx = \left[-e^{-x}\right]_1^t = -e^{-t} + e^{-1}\,,$$

hence

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \left(-e^{-t} + e^{-1}\right) = e^{-1}\,,$$

so $\int_1^\infty e^{-x}\,dx$ is convergent. Hence, by the comparison theorem $\int_1^\infty e^{-x^2}\,dx$ is convergent, QED.

CHAPTER 2

Applications of Integration

2.1. More about Areas

2.1.1. Area Between Two Curves. The area between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ (f, g continuous and $f(x) \geq g(x)$ for x in $[a, b]$) is

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \boxed{\int_a^b [f(x) - g(x)]\,dx}\,.$$

Calling $y_T = f(x)$, $y_B = g(x)$, we have:

$$\boxed{A = \int_a^b (y_T - y_B)\,dx}$$

Example: Find the area between $y = e^x$ and $y = x$ bounded on the sides by $x = 0$ and $x = 1$.

Answer: First note that $e^x \geq x$ for $0 \leq x \leq 1$. So:

$$A = \int_0^1 (e^x - x)\,dx = \left[e^x - \frac{x^2}{2}\right]_0^1 = \left(e^1 - \frac{1^2}{2}\right) - \left(e^0 - \frac{0^2}{2}\right)$$
$$= e - \tfrac{1}{2} - 1 = \boxed{e - \tfrac{3}{2}}\,.$$

The area between two curves $y = f(x)$ and $y = g(x)$ that intersect at two points can be computed in the following way. First find the intersection points a and b by solving the equation $f(x) = g(x)$. Then find the difference:

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx\,.$$

If the result is negative that means that we have subtracted wrong. Just take the result in absolute value.

Example: Find the area between $y = x^2$ and $y = 2 - x$. *Solution*: First, find the intersection points by solving $x^2-(2-x) = x^2+x-2 = 0$. We get $x = -2$ and $x = 1$. Next compute:

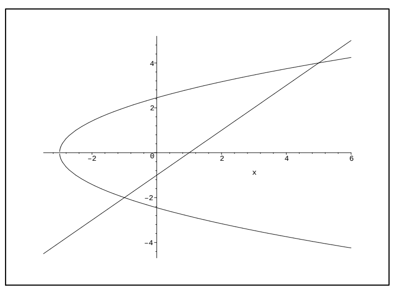
$$\int_{-2}^{1} (x^2 - (2 - x))\, dx = \int_{-2}^{1} (x^2 + x - 2)\, dx = -9/2\,.$$

Hence the area is 9/2.

Sometimes it is easier or more convenient to write x as a function of y and integrate respect to y. If $x_L(y) \le x_R(y)$ for $p \le y \le q$, then the area between the graphs of $x = x_L(y)$ and $x = x_R(y)$ and the horizontal lines $y = p$ and $y = q$ is:

$$\boxed{A = \int_{p}^{q} (x_R - x_L)\, dy}$$

Example: Find the area between the line $y = x-1$ and the parabola $y^2 = 2x + 6$.



Answer: The intersection points between those curves are $(-1, -2)$ and $(5, 4)$, but in the figure we can see that the region extends to the left of $x = -1$. In this case it is easier to write

$$x_L = \tfrac{1}{2}y^2 - 3\,, \qquad x_R = y + 1\,,$$

and integrate from $y = -2$ to $y = 4$:

$$\begin{aligned} A = \int_{-2}^{4} (x_R - x_L)\, dx &= \int_{-2}^{4} \left\{(y + 1) - (\tfrac{1}{2}y^2 - 3)\right\}\, dx \\ &= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right)\, dx \\ &= \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right]_{-2}^{4} \\ &= \boxed{18} \end{aligned}$$

2.2. Volumes

2.2.1. Volumes by Slices. First we study how to find the volume of some solids by the method of cross sections (or "slices"). The idea is to divide the solid into slices perpendicular to a given reference line. The volume of the solid is the sum of the volumes of its slices.

2.2.2. Volume of Cylinders. A cylinder is a solid whose cross sections are parallel translations of one another. The volume of a cylinder is the product of its height and the area of its base:

$$V = Ah\,.$$

2.2.3. Volume by Cross Sections. Let R be a solid lying alongside some interval $[a, b]$ of the x-axis. For each x in $[a, b]$ we denote $A(x)$ the area of the cross section of the solid by a plane perpendicular to the x-axis at x. We divide the interval into n subintervals $[x_{i-1}, x_i]$, of length $\Delta x = (b - a)/n$ each. The planes that are perpendicular to the x-axis at the points $x_0, x_1, x_2, \dots, x_n$ divide the solid into n slices. If the cross section of R changes little along a subinterval $[x_{i-1}, x_i]$, the slab positioned alongside that subinterval can be considered a cylinder of height Δx and whose base equals the cross section $A(x_i^*)$ at some point x_i^* in $[x_{i-1}, x_i]$. So the volume of the slice is

$$\Delta V_i \approx A(x_i^*)\,\Delta x\,.$$

The total volume of the solid is

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x\,.$$

Once again we recognize a Riemann sum at the right. In the limit as $n \to \infty$ we get the so called *Cavalieri's principle*:

$$V = \int_a^b A(x)\,dx\,.$$

Of course, the formula can be applied to any axis. For instance if a solid lies alongside some interval $[a, b]$ on the y axis, the formula becomes

$$V = \int_a^b A(y)\,dy\,.$$

Example: Find the volume of a cone of radius r and height h.

Answer: Assume that the cone is placed with its vertex in the origin of coordinates and its axis on the x-axis. The x coordinate runs through the interval $[0, h]$. The cross section of the cone at each point x is a circular disk of radius xr/h, hence its area is $A(x) = \pi(xr/h)^2 = \pi r^2 x^2/h^2$. The volume of the cone can now be computed by Cavalieri's formula:

$$V = \int_0^h \frac{\pi r^2}{h^2}\, x^2\, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\,\frac{h^3}{3} = \frac{1}{3}\pi r^2 h\,.$$

2.2.4. Solids of Revolution. Consider the plane region between the graph of the function $y = f(x)$ and the x-axis along the interval $[a, b]$. By revolving that region around the x-axis we get a *solid of revolution*. Now each cross section is a circular disk of radius y, so its area is $A(x) = \pi y^2 = \pi[f(x)]^2$. Hence, the volume of the solid is

$$V = \int_a^b \pi y^2\, dx = \int_a^b \pi[f(x)]^2\, dx\,.$$

Example: Find the volume of a cone of radius r and height h.

Answer: Assume that the cone is placed with its vertex in the origin of coordinates and its axis on the x-axis. This cone can be obtained by revolving the area under the line $y = rx/h$ between $x = 0$ and $x = h$ around the x-axis. So its volume is

$$V = \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx = \int_0^h \frac{\pi r^2}{h^2}\, x^2\, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\,\frac{h^3}{3} = \frac{1}{3}\pi r^2 h\,.$$

If the revolution is performed around the y-axis, the roles of x and y are interchanged, so in that case the formula is

$$V = \int_a^b \pi x^2\, dy\,,$$

where x must be written as a function of y.

If the region being revolved is the area between two curves $y = f(x)$ and $y = g(x)$, then each cross section is an annular ring (or washer)with outer radius $f(x)$ and inner radius $g(x)$ (assuming $f(x) \geq g(x) \geq 0$.) The area of the annular ring is $A(x) = \pi(f(x)^2 - g(x)^2)$, hence the volume of the solid will be:

$$V = \int_a^b \pi\, \left[(y_T)^2 - (y_B)^2\right]\, dx = \int_a^b \pi\, \left[f(x)^2 - g(x)^2\right]\, dx\,.$$

If the revolution is performed around the y-axis, then:

$$V = \int_a^b \pi \, \left[(x_R)^2 - (x_L)^2\right] \, dy \,.$$

Example: Find the volume of the solid obtained by revolving the area between $y = x^2$ and $y = \sqrt{x}$ around the x-axis.

Solution: First we need to find the intersection points of these curves in order to find the interval of integration:

$$\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases} \quad \Rightarrow \quad (x,y) = (0,0) \quad \text{and} \quad (x,y) = (1,1)\,,$$

hence we must integrate from $x = 0$ to $x = 1$:

$$\begin{aligned} V &= \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] \, dx = \pi \int_0^1 \left(x - x^4\right) \, dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10} \,. \end{aligned}$$

2.2.5. Volumes by Shells. Next we study how to find the volume of some solids by the method of shells. Now the idea is to divide the solid into shells and add up their volumes.

2.2.6. Volume of a Cylindrical Shell. A cylindrical shell is the region between two concentric circular cylinders of the same height h. If their radii are r_1 and r_2 respectively, then the volume is:

$$V = \pi r_2^2 h - \pi r_1^2 h = \pi h (r_2^2 - r_1^2) = \pi h \overbrace{(r_2 + r_1)}^{2\bar{r}} \overbrace{(r_2 - r_1)}^{t} = 2\pi \bar{r} t h \,,$$

where $\bar{r} = (r_2 + r_1)/2$ is the average radius, and $t = r_2 - r_1$ is the thickness of the shell.

2.2.7. Volumes by Cylindrical Shells. Consider the solid generated by revolving around the y-axis the region under the graph of $y = f(x)$ between $x = a$ and $x = b$. We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of length $\Delta x = (b - a)/n$ each. The volume V of the solid is the sum of the volumes ΔV_i of the shells determined by the partition. Each shell, obtained by revolving the region under $y = f(x)$ over the subinterval $[x_{i-1}, x_i]$, is approximately cylindrical. Its height is $f(\overline{x}_i)$, where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$. Its thickness is Δx. Its average radius is $\overline{x}_i$. Hence its volume is

$$\Delta V_i \approx 2\pi \overline{x}_i f(\overline{x}_i)\Delta x\,,$$

and the volume of the solid is

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i)\Delta x\,.$$

As $n \to \infty$ the right Riemann sum converges to the following integral:

$$V = \int_a^b 2\pi x f(x)\,dx = \int_a^b 2\pi xy\,dx\,.$$

Example: Find the volume of the solid obtained by revolving around the y-axis the plane area between the graph of $y = 1-x^2$ and the x-axis.

Answer: The graph intersects the positive x-axis at $x = 1$, so the interval is $[0, 1]$. Hence

$$\begin{aligned} V = \int_0^1 2\pi xy\,dx = \int_0^1 2\pi x\cdot(1-x^2)\,dx = 2\pi\int_0^1 (x-x^3)\,dx \\ = 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}\,. \end{aligned}$$

2.2.8. Revolving the Region Between Two Curves. Here we find the volume of the solid obtained by revolving around the y-axis the area between two curves $y = f(x)$ and $y = g(x)$ over an interval $[a, b]$. The computation is similar, but if $f(x) \geq g(x)$ the shells will have height $f(x_i^*) - g(x_i^*)$, so the volume will be given by the integral:

$$V = \int_a^b 2\pi x(f(x) - g(x))\,dx = \int_a^b 2\pi x(y_T - y_B)\,dx\,.$$

Example: Find the volume of the solid obtained by revolving the plane region limited by the curves $y = x$ and $y = x^2$ over the interval $[0, 1]$.

Answer: In $[0, 1]$ we have $x \geq x^2$, so:

$$\begin{aligned} V &= \int_0^1 2\pi x\,(y_T - y_B)\,dx = 2\pi\int_0^1 x\,(x-x^2)\,dx \\ &= 2\pi\int_0^1 (x^2-x^3)\,dx = 2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = 2\pi\frac{1}{12} = \frac{\pi}{6}\,. \end{aligned}$$

If the region is revolved around the x-axis then the variables x and y reverse their roles:

$$V = \int_a^b 2\pi y\,(x_R - x_L)\,dy\,.$$

2.2.9. Revolving Around an Arbitrary Line. If the plane region is revolved around a vertical line $y = c$, the radius of the shell will be $x - c$ (or $c - x$, whichever is positive) instead of x, so the formula becomes:

$$V = \int_a^b 2\pi(x-c)(f(x)-g(x))\,dx = \int_a^b 2\pi(x-c)(y_T - y_B)\,dx\,.$$

Similarly, if the region is revolved around the horizontal line $x = c$, the formula becomes:

$$V = \int_a^b 2\pi(y-c)(f(y)-g(y))\,dy = \int_a^b 2\pi(y-c)(x_R - x_L)\,dy\,,$$

where $y - c$ must be replaced by $c - y$ if $c > y$.

2.3. Arc Length, Parametric Curves

2.3.1. Parametric Curves. A parametric curve can be thought of as the trajectory of a point that moves trough the plane with coordinates $(x, y) = (f(t), g(t))$, where $f(t)$ and $g(t)$ are functions of the *parameter* t. For each value of t we get a point of the curve. Example: A parametric equation for a circle of radius 1 and center $(0, 0)$ is:

$$x = \cos t, \qquad y = \sin t\,.$$

The equations $x = f(t)$, $y = g(t)$ are called *parametric equations*.

Given a parametric curve, sometimes we can eliminate t and obtain an equivalent non-parametric equation for the same curve. For instance t can be eliminated from $x = \cos t$, $y = \sin t$ by using the trigonometric relation $\cos^2 t + \sin^2 t = 1$, which yields the (non-parametric) equation for a circle of radius 1 and center $(0, 0)$:

$$x^2 + y^2 = 1\,.$$

Example: Find a non-parametric equation for the following parametric curve:

$$x = t^2 - 2t, \qquad y = t + 1\,.$$

Answer: We eliminate t by isolating it from the second equation:

$$t = (y - 1)\,,$$

and plugging it in the first equation:

$$x = (y - 1)^2 - 2(y - 1)\,.$$

i.e.:

$$x = y^2 - 4y + 3\,,$$

which is a parabola with horizontal axis.

2.3.2. Arc Length. Here we describe how to find the length of a *smooth arc*. A smooth arc is the graph of a continuous function whose derivative is also continuous (so it does not have corner points).

If the arc is just a straight line between two points of coordinates (x_1, y_1), (x_2, y_2), its length can be found by the Pythagorean theorem:

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}\,,$$

where $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$.

More generally, we approximate the length of the arc by inscribing a polygonal arc (made up of straight line segments) and adding up the lengths of the segments. Assume that the arc is given by the parametric functions $x = f(x)$, $y = g(x)$, $a \le t \le b$.

We divide the interval into n subintervals of equal length. The corresponding points in the arc have coordinates $(f(t_i), g(t_i))$, so two consecutive points are separated by a distance equal to

$$L_i = \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2}\,.$$

We have $\Delta t = t_i - t_{i-1} = (b-a)/n$. On the other hand, by the mean value theorem

$$\begin{aligned} f(t_i) - f(t_{i-1}) &= f'(t_i^*)\,\Delta t \\ g(t_i) - f(t_{i-1}) &= g'(t_i^*)\,\Delta t \end{aligned}$$

for some t_i^* in $[t_{i-1}, t_i]$. Hence

$$L_i = \sqrt{[f'(x_i^*)\,\Delta t]^2 + [g'(t_i^*)\,\Delta t]^2} = \sqrt{[f'(t_i^*)]^2 + [g'(t_i^*)]^2}\,\Delta t\,.$$

The total length of the arc is

$$L \approx \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^*)]^2}\,\Delta t\,,$$

which converges to the following integral as $n \to \infty$:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\;dt\,.$$

This formula can also be expressed in the following (easier to remember) way:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt}$$

The last formula can be obtained by integrating the length of an "infinitesimal" piece of arc

$$ds = \sqrt{(dx)^2 + (dy)^2} = dt\,\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.$$

Example: Find the arc length of the curve $x = t^2$, $y = t^3$ between $(1, 1)$ and $(4, 8)$.

Answer: The given points correspond to the values $t = 1$ and $t = 2$ of the parameter, so:

$$\begin{aligned}
L &= \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt \\
&= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt \\
&= \int_1^2 t\sqrt{4 + 9t^2}\, dt \\
&= \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du \qquad\qquad (u = 4 + 9t^2) \\
&= \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \\
&= \boxed{\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})}\,.
\end{aligned}$$

In cases when the arc is given by an equation of the form $y = f(x)$ or $x = f(x)$ the formula becomes:

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx}$$

or

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy}$$

Example: Find the length of the arc defined by the curve $y = x^{3/2}$ between the points $(0, 0)$ and $(1, 1)$.

Answer:

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+[(x^{3/2})']^2}\,dx \\
&= \int_0^1 \sqrt{1+\left(\frac{3x^{1/2}}{2}\right)^2}\,dx = \int_0^1 \sqrt{1+\frac{9x}{4}}\,dx \\
&= \left[\frac{1}{27}(4+9x)^{3/2}\right]_0^1 = \boxed{\frac{1}{27}(13^{3/2}-8)}.
\end{aligned}$$

2.4. Average Value of a Function (Mean Value Theorem)

2.4.1. Average Value of a Function. The average value of finitely many numbers $y_1, y_2, \dots, y_n$ is defined as

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}\,.$$

The average value has the property that if each of the numbers $y_1, y_2, \dots, y_n$ is replaced by y_{ave}, their sum remains the same:

$$y_1 + y_2 + \cdots + y_n = \overbrace{y_{\text{ave}} + y_{\text{ave}} + \cdots + y_{\text{ave}}}^{(n \text{ times})}$$

Analogously, the average value of a function $y = f(x)$ in the interval $[a, b]$ can be defined as the value of a constant f_{ave} whose integral over $[a, b]$ equals the integral of $f(x)$:

$$\int_a^b f(x)\,dx = \int_a^b f_{\text{ave}}\,dx = (b - a)\,f_{\text{ave}}\,.$$

Hence:

$$\boxed{f_{\text{ave}} = \frac{1}{b - a}\int_a^b f(x)\,dx}\,.$$

2.4.2. The Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b - a}\int_a^b f(x)\,dx\,,$$

i.e.,

$$\int_a^b f(x)\,dx = f(c)(b - a)\,.$$

Example: Assume that in a certain city the temperature (in °F) t hours after 9 A.M. is represented by the function

$$T(t) = 50 + 14\sin\frac{\pi t}{12}\,.$$

Find the average temperature in that city during the period from 9 A.M. to 9 P.M.

Answer:

$$\begin{aligned}
T_{\text{ave}} &= \frac{1}{12-0}\int_0^{12}\left(50 + 14\sin\frac{\pi t}{12}\right)\,dt \\
&= \frac{1}{12}\left[50t - \frac{14\cdot 12}{\pi}\cos\frac{\pi t}{12}\right]_0^{12} \\
&= \frac{1}{12}\left\{\left(50\cdot 12 - \frac{168}{\pi}\cos\frac{12\pi}{12}\right) - \left(50\cdot 0 - \frac{168}{\pi}\cos 0\right)\right\} \\
&= 50 + \frac{28}{\pi} \approx 58.9\,.
\end{aligned}$$

2.5. Applications to Physics and Engineering

2.5.1. Work. Work is the energy produced by a force F pushing a body along a given distance d. If the force is constant, the work done is the product

$$W = F \cdot d\,.$$

The SI (international) unit of work is the joule (J), which is the work done by a force of one Newton (N) pushing a body along one meter (m). In the American system a unit of work is the foot-pound. Since 1 N = 0.224809 lb and 1 m = 3.28084 ft, we have 1 J = 0.737561 ft lb.

More generally, assume that the force is variable and depends on the position. Let $F(x)$ be the force function. Assume that the force pushes a body from a point $x = a$ to another point $x = b$. In order to find the total work done by the force we divide the interval $[a, b]$ into small subintervals $[x_{i-1}, x_i]$ so that the change of $F(x)$ is small along each subinterval. Then the work done by the force in moving the body from x_{i-1} to x_i is approximately:

$$\Delta W_i \approx F(x_i^*)\,\Delta x\,,$$

where $\Delta x = x_i - x_{i-1} = (b-a)/n$ and x_i^* is any point in $[x_{i-1}, x_i]$. So, the total work is

$$W = \sum_{i=1}^{n} \Delta W_i \approx \sum_{i=1}^{n} F(x_i^*)\,\Delta x\,.$$

As $n \to \infty$ the Riemann sum at the right converges to the following integral:

$$W = \int_a^b F(x)\,dx\,.$$

2.5.2. Elastic Springs. Consider a spring on the x-axis so that its right end is at $x = 0$ when the spring is at its rest position. According to *Hook's Law*, the force needed to stretch the spring from 0 to x is proportional to x, i.e.:

$$F(x) = kx\,,$$

where k is the so called *spring constant*.

The energy needed to stretch the spring from 0 to a is then the integral

$$W = \int_0^a kx\,dx = k\frac{a^2}{2}\,.$$

2.5.3. Work Done Against Gravity. According to Newton's Law, the force of gravity at a distance r from the center of the Earth is

$$F(r) = \frac{k}{r^2}\,,$$

where k is some positive constant.

The energy needed to lift a body from a point at distance R_1 from the center of the Earth to another point at distance R_2 is given by the following integral

$$W = \int_{R_1}^{R_2} \frac{k}{r^2}\,dr = \left[-\frac{k}{r}\right]_{R_1}^{R_2} = k\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Example: Find the energy needed to lift 1000 Km a body whose weight is 1 N at the surface of Earth. The Earth radius is 6378 Km.

Answer: First we must determine the value of the constant k in this case. Since the weight of the body for $r = 6378$ Km is 1 N we have $k/6378^2 = 1$, so $k = 6378^2$. Next we have $R_1 = 6378$, $R_2 = 6378 + 1000 = 7378$, hence

$$W = 6378^2\left(\frac{1}{6378} - \frac{1}{7378}\right) = 864.462 \text{ N Km}\,.$$

Since 1 Km = 1000 m, the final result in joule is

$$864.462 \text{ N Km} = 864.462 \text{ N} \times 1000 \text{ m} = 864462 \text{ J}\,.$$

2.5.4. Work Done Filling a Tank. Consider a tank whose bottom is at some height $y = a$ and its top is at $y = b$. Assume that the area of its cross section is $A(y)$. We fill the tank by lifting from the ground ($y = 0$) tiny layers of thickness dy each. Their mass is $\rho A(y)\,dy$, where ρ is the density (mass per unit of volume) of the liquid that we are putting in the tank. We get their weight dF by multiplying by the acceleration of gravity g ($= 9.8$ m/s^2 $= 32$ ft/s^2), so $dF = \rho g A(y)\,dy$. The work needed to lift each layer is

$$dW = dF \cdot y = \rho g y A(y)\,dy\,.$$

Hence, the work needed to fill the tank is

$$W = \int_a^b \rho g y A(y)\,dy\,.$$

2.5.5. Emptying a Tank. Consider a tank like the one in the previous paragraph. Now we empty it by pumping its liquid to a fix height h. The analysis of the problem is similar to the previous paragraph, but now the work done to pump a tiny layer of thickness dy is

$$dW = dF \cdot (h - y) = \rho g\,(h - y) A(y)\,dy\,.$$

Hence the total work needed to empty the tank is

$$W = \int_a^b \rho g\,(h - y) A(y)\,dy\,.$$

2.5.6. Force Exerted by a Liquid Against a Vertical Wall. The pressure p of an homogeneous liquid of density ρ at depth h is

$$p = \rho g h\,.$$

When the pressure is constant, the force exerted by the liquid against a surface is the product of the pressure and the area of the surface. However the pressure against a vertical wall is not constant because it depends on the depth.

Assume that the surface of the liquid is at $y = c$ and we place a vertical plate of width $w(y)$ between $y = a$ and $y = b$. The force exerted at y (so at depth $h = c - y$) against a small horizontal strip of height dy and width $w(y)$ (area $= w(y)\,dy$) is

$$dF = \rho g\,(c - y) w(y)\,dy\,.$$

hence the total force is

$$F = \int_a^b \rho g\,(c - y) w(y)\,dy\,.$$

Example: A cylindrical tank of radius 1 m and full of water ($\rho =$ 1000 Kg/m^3, $g = 9.8$m/s^2) is lying on its side. What is the pressure exerted by the water on its (vertical) bottom?

Answer: We assume the center of the tank is at $y = 0$, so the top of the liquid is at $y = 1$, and its bottom is at $y = -1$. On the other hand we obtain geometrically $w(y) = 2\sqrt{1 - y^2}$. Hence the total force is:

$$F = \int_{-1}^{1} 1000 \cdot 9.8 \cdot (1 - y) 2\sqrt{1 - y^2}\,dy = 19800 \cdot \frac{\pi}{2} = 30787.6 \text{ N}\,.$$

2.6. Probability

2.6.1. Continuous Random Variables. A *random variable* is a real-valued function defined on some set of possible outcomes of a random experiment; e.g. the number of points obtained after rolling a dice - which can be 1, 2, 3, 4, 5 or 6. In this example the random variable can take only a discrete set of values. If the variable can take a continuous set of values then it is called a *continuous random variable*, e.g. a person's height.

Given a random variable X, its *probability distribution function* is the function $F(x) = P(X \leq x) =$ probability that the random variable X takes a value less than or equal to x. For instance if $F(x)$ is the probability distribution function of the number of points obtained after rolling a dice, then $F(4.7) = P(X \leq 4.7) =$ probability that the number of points is less than or equal to 4.7, i.e., the number of points is 1, 2, 3 or 4, so the probability is $4/6 = 2/3$, and $F(4.7) = 2/3$.

If the random variable is continuous then we can also define a *probability density function* $f(x)$ equal to the limit as $\Delta x \to 0$ of the probability that the random variable takes a value in a small interval of length Δx around x divided by the length of the interval. This definition means that $f(x) = F'(x)$. The probability that the random variable takes a value in some interval $[a, b]$ is

$$P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)\, dx\,.$$

In general the probability density of a random variable satisfies two conditions:

(1) $f(x) \geq 0$ for every x (probabilities are always non-negative).
(2) $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$ (the probability of a sure event is 1).

Example: A probability distribution is called *uniform* on a set S if its probability density is constant on S. Find the probability density of the uniform distribution on the interval $[2, 5]$.

Answer: The probability density function must be constant on $[2, 5]$, so for $2 \leq x \leq 5$ we have $f(x) = c$ for some constant c. On the other hand $f(x) = 0$ for x outside $[2, 5]$, hence:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{5} c\,dx = c(5-2) = 3c\,,$$

so $c = 1/3$. Hence

$$f(x) = \begin{cases} 1/3 & \text{if } 2 \leq x \leq 5, \\ 0 & \text{otherwise.} \end{cases}$$

2.6.2. Means. The *mean* or *average* of a discrete random variable that takes values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively is

$$\overline{x} = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n = \sum_{i=1}^{n} x_i p_i\,.$$

For instance the mean value of the points obtained by rolling a dice is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2} = 3.5\,.$$

This means that if we roll the dice many times in average we may expect to get about 3.5 points per roll.

For continuous random variables the probability is replaced with the probability density function, and the sum becomes an integral:

$$\mu = \overline{x} = \int_{-\infty}^{\infty} x f(x)\,dx\,.$$

2.6.3. Waiting Times. The time that we must wait for some event to occur (such as receiving a telephone call) can be modeled with a random variable of density

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ ce^{-ct} & \text{if } t \geq 0, \end{cases}$$

were c is a positive constant. Note that, as expected:

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{0}^{\infty} ce^{-ct}\,dx = \left[-e^{-ct}\right]_{0}^{\infty} = \lim_{u\to\infty} \{-e^{-cu} - (-e^{0})\} = 1\,.$$

The mean waiting time can be computed like this:

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} t f(t)\, dt \\
&= \int_{0}^{\infty} tce^{-ct}\, dx \\
&= \left[-te^{-ct}\right]_0^{\infty} + \int_0^{\infty} e^{-ct}\, dx \quad \text{(I. by parts)} \\
&= 0 + \left[-\frac{e^{-ct}}{c}\right]_0^{\infty} \\
&= \frac{1}{c}\,,
\end{aligned}$$

hence $\mu = 1/c$. So we can rewrite the density function like this:

$$f(x) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{\mu} e^{-t/\mu} & \text{if } t \geq 0, \end{cases}$$

Example: Assume that the average waiting time for a catastrophic meteorite to strike the Earth is 100 million years. Find the probability that the Earth will suffer a catastrophic meteorite impact in the next 100 years. Find the probability that no such catastrophic event will happen in the next 5 billion years.

Answer: We have $\mu = 10^8$ years, so the probability density function is

$$f(t) = 10^{-8} e^{-10^{-8}\, t} \qquad (t \geq 0)\,.$$

So the answer to the first question is

$$\int_0^{100} 10^{-8} e^{-10^{-8}\, t}\, dt = \left[1 - e^{-10^{-8}\, t}\right]_0^{100} = 1 - e^{-10^{-8} \cdot 100} = 1 - e^{-10^{-6}} \approx 10^{-6}\,,$$

i.e., about 1 in a million. Regarding the second question, the probability is

$$\begin{aligned}
1 - \int_0^{5\cdot 10^9} 10^{-8} e^{-10^{-8}\, t}\, dt &= 1 - \left[1 - e^{-10^{-8}\, t}\right]_0^{5\cdot 10^9} \\
&= 1 - \left\{1 - e^{-10^{-8} \cdot 5 \cdot 10^9}\right\} = e^{-50} \approx 2 \cdot 10^{-22}\,,
\end{aligned}$$

which is practically zero.

2.6.4. Normal Distributions. Many important phenomena follow a so called *Normal Distribution*, whose density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} .$$

Its mean is μ. The positive constant σ is called *standard deviation*; it measures how spread out the values of the random variable are.

Example: Intelligent Quotient (IQ) scores are distributed normally with mean $\mu = 100$ and standard deviation $\sigma = 15$. What proportion of the population has an IQ between 70 and 130?

Answer: The integral cannot be evaluated in terms of elementary functions, but it can be approximated with numerical methods:

$$P(70 \le X \le 130) = \int_{70}^{130} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/2\cdot 15^2}\, dx \approx \boxed{0.9544997360\cdots} .$$

Another approach for solving these kinds of problems is to use the *error function*, defined in the following way:

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt .$$

The following are some of its values (rounded to three decimal places):

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\phi(x)$	0.0	0.112	0.223	0.329	0.438	0.520	0.604	0.678	0.742	0.797
x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\phi(x)$	0.843	0.880	0.910	0.934	0.952	0.966	0.976	0.984	0.989	0.993
x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
$\phi(x)$	0.995	0.997	0.998	0.999	0.999	1.000	1.000	1.000	1.000	1.000

Example: Solve the previous problem using the error function.

Answer: We need to transform our integral into another expression containing the error function:

$$
\begin{aligned}
P(70 \leq X \leq 130) &= \int_{70}^{130} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-u^2}\,du && [u = (x-100)/15\sqrt{2}] \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{2}} e^{-u^2}\,du && \text{(by symmetry)} \\
&= \phi(\sqrt{2}) \approx \phi(1.4) \approx \boxed{0.952}
\end{aligned}
$$

CHAPTER 3

Differential Equations

3.1. Differential Equations and Separable Equations

3.1.1. Population Growth. The growth of a population is usually modeled with an equation of the form

$$\frac{dP}{dt} = kP\,,$$

where P represents the number of individuals an a given time t. This model assumes that the rate of growth of population is proportional to the population size.

A solution to this equation is the exponential function:

$$P(t) = Ce^{kt}\,.$$

Check: $P'(t) = kCe^{kt} = kP(t)$.

A more realistic model takes into account that any environment has a limited *carrying capacity* K, so if P reaches K the population stops growing. The model in this case is the following:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)\,.$$

This is called the *logistic differential equation.*

3.1.2. Motion of a Spring. Consider an object of mass m at the end of a vertical spring. According to Hook's law the restoring force of a spring stretched (or compressed) a distance x from its natural length is

$$F = -kx\,,$$

where k is a positive constant (the *spring constant*) and the negative sign expresses that the sense of the force is opposite to the sense of the stretching. By Newton's Second Law (force equals mass times acceleration):

$$m \frac{d^2x}{dt^2} = -kx \, ,$$

or equivalently:

$$\frac{d^2x}{dt^2} = -\frac{k}{m} x \, .$$

This is an example of a *second order differential equation* because it involves second order derivatives.

3.1.3. General Differential Equations.

A *differential equation* is an equation that contains one or more unknown functions and one or more of its derivatives. The *order* of the differential equation is the order of the highest derivative that occurs in the equation.

3.1.4. First-order Differential Equations.

A *first-order differential equation* is an equation of the form

$$\frac{dy}{dx} = F(x, y) \, ,$$

where $F(x, y)$ is a function of x and y. A *solution* of the differential equation is a function $y(x)$ such that $y'(x) = F(x, y(x))$ for all x in some appropriate interval.

Example: Consider the following differential equation:

$$\frac{dy}{dx} = \frac{2y}{x} \, .$$

A possible solution for that equation is, for instance, $y = x^2$, because

$$\frac{dy}{dx} = y'(x) = (x^2)' = 2x \, ,$$

and

$$2 \, \frac{y}{x} = 2 \, \frac{x^2}{x} = 2x \, ,$$

hence $y'(x) = \dfrac{2 \, y(x)}{x}$ for all $x \neq 0$.

3.1.5. Separable Differential Equations. A differential equation is said to be *separable* if it can be written in the form

$$f(y)\,dy = g(x)\,dx\,,$$

so that the left hand side depends on y only and the right hand side depends on x only. In particular this is true if the equation is of the form

$$\frac{dy}{dx} = g(x)\,\phi(y)\,,$$

where the right hand side is a product of a function of x and a function of y. In this case we get:

$$\frac{1}{\phi(y)}\,dy = g(x)\,dx\,.$$

Given the equation

$$f(y)\,dy = g(x)\,dx\,,$$

we can solve it by integrating both sides. Since the antiderivatives of a function differ in a constant, we get:

$$\int f(y)\,dy = \int g(x)\,dx + C\,,$$

If $F(y) = \int f(y)\,dy$ and $G(x) = \int g(x)\,dx$ then the solution takes the form

$$F(y) = G(x) + C\,.$$

Next we will try to solve this equation algebraically in order to either write y as a function of x, or x as a function of y.

Example: Consider the equation

$$\frac{dy}{dx} = y^2\,x\,.$$

The right hand side is the product of a function of x and a function of y, so it is separable:

$$\frac{1}{y^2}\,dy = x\,dx\,.$$

Integrating both sides we get:

$$-\frac{1}{y} = \frac{x^2}{2} + C\,,$$

hence

$$y = -\frac{2}{x^2 + 2C} = -\frac{2}{x^2 + C'}\,,$$

where C' is a new constant equal to $2C$.

3.1.6. Initial Value Problems. A differential equation together with an initial condition

$$\begin{cases} \dfrac{dy}{dx} = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

is called an *initial value problem.*

The initial condition can be used to determine the value of the constant in the solution of the equation.

Example: Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = y^2\, x \\ y(0) = 1 \end{cases}$$

Solution: We already found the general solution to the differential equation:

$$y = -\frac{2}{x^2 + C}\,.$$

Next we let $x = 0$ and $y = 1$, and solve for C:

$$1 = -\frac{2}{C} \quad \Longrightarrow \quad C = -2\,.$$

So the solution is

$$y = -\frac{2}{x^2 - 2} = \frac{2}{2 - x^2}\,.$$

3.2. Directional Fields and Euler's Method

Here we study a graphical method (direction fields) and a numerical method (Euler's method) to solve differential equations.

3.2.1. Slope Fields. Consider a differential equation of the form

$$\frac{dy}{dx} = F(x, y)\,.$$

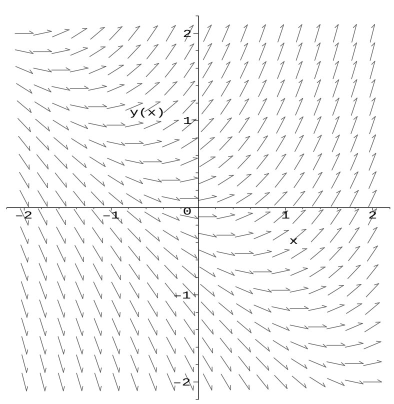
If $y(x)$ is a solution, the slope of its graph at each point $(x, y(x))$ should be equal to $F(x, y)$. So the right hand side of the equation can be interpreted as a *slope field* in the xy-plane. The graph of a solution is called a *solution curve* for the slope field. Each solution curve is a *particular solution* of the slope field. A point (x_0, y_0) in the xy-plane plays the role of an *initial condition*, and the solution curve that passes through that point corresponds to the solution of the differential equation satisfying the corresponding initial condition $y(x_0) = y_0$.

3.2.2. Direction Fields. This method consists of interpreting the differential equation as a slope field and sketch solutions just by following the field.

Example: The direction field for the differential equation

$$y' = x + y$$

looks like this:



3.2.3. Euler's Method. Euler's method consists of approximating solutions to a differential equation with polygonal lines made of short straight lines each with slope equal to y' at their initial point. So assume that we want to find approximate values of a solution for an initial-value problem

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

at equally spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, $\dots$, where h is called the *step size*. We take (x_0, y_0) as the initial point of the solution. The slope at (x_0, y_0) is $y_0' = F(x_0, y_0)$, hence next point will be (x_1, y_1) so that $(y_1 - y_0)/h = F(x_0, y_0)$, i.e., $y_1 = y_0 + hF(x_0, y_0)$. Proceeding in the same way we get in general:

$$\begin{aligned} y_1 &= y_0 + hF(x_0, y_0) \\ y_2 &= y_1 + hF(x_1, y_1) \\ &\dots \\ y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1}) \\ &\dots \end{aligned}$$

Example: Use Euler's method with step size 0.1 to find approximate values of the solution to the initial value problem

$$\begin{cases} y' = x + y \\ y(0) = 1 \end{cases}$$

Answer: We have:

$$\begin{aligned} &y(0) = y_0 = 1 \\ &y(0.1) \approx y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1 \\ &y(0.2) \approx y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22 \\ &y(0.3) \approx y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362 \\ &\qquad \dots \end{aligned}$$

3.3. Exponential Growth and Decay

3.3.1. Natural Growth. Consider population with $P(t)$ individuals at time t and with constant *birth rate* β (births per unit of time) and *death* rate δ (deaths per unit of time). This basically means that if P does not change, then during a unit of time (say, a year), βP births and δP deaths will occur. Since P in fact varies, we need to use smaller intervals of time $[t, t + \Delta t]$ in which P can be considered almost constant. During such interval of time the number of births will be $\beta P \Delta t$, and the number of deaths $\delta P \Delta t$. So the change in the population will be

$$\Delta P = P(t + \Delta t) - P(t) \approx \beta P \Delta t - \delta P \Delta t\,.$$

Dividing by Δt and finding the limit as $\Delta t \to 0$ we get

$$P'(t) = (\beta - \delta) P(t)\,,$$

i.e.,

$$\frac{dP}{dt} = kP\,,$$

where $k = \beta - \delta$. With $x(t)$ in place of $P(t)$ we get the *Natural Growth Equation*:

$$\frac{dx}{dt} = kx\,.$$

This equation can be solved by separation of variables:

$$\begin{aligned} \frac{dx}{x} &= k\,dt \\ \int \frac{dx}{x} &= \int k\,dt + C \\ \ln x &= kt + C \\ x &= e^{kt+C} = Ae^{kt}\,, \end{aligned}$$

where $A = e^C$. Putting $t = 0$ we see that $A = x_0 = x(0)$, hence:

$$x(t) = x_0 e^{kt}\,.$$

Example: The current (year 2000) population of the Earth is 6 billion people, and the yearly birth and death rates are $\beta = 0.021$ and $\delta = 0.009$ respectively. Assuming the birth and death rates remain constant, find the population of the Earth in the year 2100.

Answer: For the purpose of the problem we can take $t = 0$ in the year 2000, so the year 2100 will correspond to $t = 100$. So we have

$x_0 = 6$ billion, and $k = \beta - \delta = 0.021 - 0.009 = 0.012$, hence:

$$P(t) = P_0\, e^{kt} = 6\, e^{0.012t}$$

in billions. So the solution is

$$P(100) = 6\, e^{0.012\times 100} = 19.92 \text{ billion}\,.$$

3.3.2. Radioactive Decay. Consider a given sample of radioactive material with $N(t)$ atoms at time t. During a given unit of time a fix fraction of these atoms will spontaneously decay, so the sample behaves like a population with a constant death rate and no births:

$$\frac{dN}{dt} = -kN\,,$$

where $k > 0$ is the *decay constant*. The solution to this equation is

$$N(t) = N_0\, e^{-kt}\,,$$

where N_o is the number of atoms at time $t = 0$.

The *half-life* τ of the material is the time required for half of the sample to decay, i.e.:

$$\frac{1}{2} N_0 = N_0\, e^{-k\tau}\,,$$

so

$$\tau = \frac{\ln 2}{k}\,.$$

3.3.3. Radiocarbon Dating. The air in the atmosphere contains two carbon isotopes: ^{12}C, which is stable, and ^{14}C, which is radioactive with a half-life of about 5700 years—so $k = \ln 2/\tau = \ln 2/5700 = 0.0001216$.

While an organism is alive, it absorbs both carbon isotopes by breathing air, so the proportion of those isotopes in living matter is the same as in air. But when an organism dies, the ^{14}C in it keeps decaying without being replaced. So by measuring the proportion of ^{14}C in an organism we can estimate for how long it has been dead.

Example: A cadaver found in an old burial site has 80% as much ^{14}C as a current day human body. When did that individual die?

Answer: We have:

$$0.80 = e^{-kt} = e^{-kt}$$

Hence

$$t = -\frac{\ln 0.80}{k} = -\frac{\ln 0.80}{0.0001216} = 1835 \text{ years ago}\,.$$

3.3.4. Continuously Compounded Interest. Consider an account opened with an initial amount of A_0 dollars and an annual interest rate r. Let $A(t)$ be the number of dollars in the account at time t. Assume the interest is compounded after an interval of time Δt. The interest produced is $rA(t)\Delta t$, so

$$A(t + \Delta t) = A(t) + rA(t)\Delta t\,,$$

i.e.

$$\frac{\Delta A}{\Delta t} = rA(t)\,.$$

The limit for $\Delta t \to 0$ is called *continuously compounded interest*. In that case we get:

$$\frac{dA}{dt} = rA(t)\,.$$

The solution to this equation is

$$A(t) = A_0 e^{rt}\,.$$

3.4. The Logistic Equation

3.4.1. The Logistic Model. In the previous section we discussed a model of population growth in which the growth rate is proportional to the size of the population. In the resulting model the population grows exponentially. In reality this model is unrealistic because environments impose limitations to population growth. A more accurate model postulates that the relative growth rate P'/P decreases when P approaches the *carrying capacity* K of the environment. The corresponding equation is the so called *logistic differential equation*:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right).$$

3.4.2. Analytic Solution. The logistic equation can be solved by separation of variables:

$$\int \frac{dP}{P(1-P/K)} = \int k\,dt\,.$$

In order to evaluate the left hand side we write:

$$\frac{1}{P(1-P/K)} = \frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P},$$

hence

$$\begin{aligned}
\int \frac{dP}{P} + \int \frac{dP}{K-P} &= \int k\,dt, \\
\ln|P| - \ln|K-P| &= kt + C, \\
\ln\left|\frac{K-P}{P}\right| &= -kt - C, \\
\left|\frac{K-P}{P}\right| &= e^{-kt-C}, \\
\frac{K-P}{P} &= Ae^{-kt} \qquad (A = \pm e^{-C})\,.
\end{aligned}$$

From here we get:

$$P = \frac{K}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{K-P_0}{P_0}\,.$$

Example: The population of the US in 1800 and 1850 was 5.3 and 23.1 million people respectively. Predict its population in 1900 and in 1950 using the exponential model of population growth. Then considering that the population of the US in 1900 was actually 76 million people correct your prediction for 1950 using the logistic model of population growth (help: with this data $k = 0.031476$ in the logistic model). What is the carrying capacity of the US according to this model?

Answer: Since we start with observations in 1800 it makes sense to choose the variable t as time elapsed since 1800. According to the exponential model the population at time t is

$$P(t) = P_0 e^{kt}\,,$$

where $P_0 = P(0)$. In our problem we have $P_0 = 5.3$. Next we determine the value of k from $P(50) = 5.3e^{k\cdot 50} = 23.1 \Rightarrow k = (\log 23.1 - \log 5.3)/50 = 0.029443$. Hence, the population at time t according to the exponential model will be $P(t) = 5.3e^{0.0294\cdot t}$, and for 1900 ($t = 100$) and 1950 ($t = 150$) we get respectively:

$$P(100) = 5.3e^{0.029443\cdot 100} = 100.7\,,$$
$$P(150) = 5.3e^{0.029443\cdot 150} = 438.8\,.$$

Now we are told that the population in 1900 was actually $P(100) = 76$ million people and are asked to correct the prediction for 1950 using the logistic model. The logistic model is given by the formula

$$P(t) = \frac{K}{1 + Ae^{-kt}}\,,$$

where $A = (K - P_0)/P_0$. The given data tell us that

$$P(50) = \frac{K}{1 + (K - 5.3)e^{-50k}/5.3} = 23.1\,,$$
$$P(100) = \frac{K}{1 + (K - 5.3)e^{-100k}/5.3} = 76\,.$$

We can obtain K and k from these system of two equations, but we are told that $k = 0.031476$, so we only need to obtain K (the carrying capacity) from one of the equations, say the first one. The result is $K = 189.4$. From here we get $A = 34.74$ and

$$P(t) = \frac{189.4}{1 + 34.74e^{-0.031476t}}\,.$$

hence in 1950, $P(150) = 144.7$ million people (the actual figure was 150.7 million people, slightly higher than expected due to the beginning of the so called "baby boom"). In this model the carrying capacity of the US is $K = 189.4$ million people.

CHAPTER 4

Infinite Sequences and Series

4.1. Sequences

A *sequence* is an infinite ordered list of numbers, for example the sequence of odd positive integers:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29 \dots$$

Symbolically the *terms* of a sequence are represented with indexed letters:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots, a_n, \dots$$

Sometimes we start a sequence with a_0 (index zero) instead of a_1.

Notation: the sequence $a_1, a_2, a_3, \dots$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Some sequences can be defined with a formula, for instance the sequence $1, 3, 5, 7, \dots$ of odd positive integers can be defined with the formula $a_n = 2n - 1$.

A *recursive definition* consists of defining the next term of a sequence as a function of previous terms. For instance the *Fibonacci sequence* starts with $f_1 = 1$, $f_2 = 1$, and then each subsequent term is the sum of the two previous ones: $f_n = f_{n-1} + f_{n-2}$; hence the sequence is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

4.1.1. Limits. The limit of a sequence is the value to which its terms approach indefinitely as n becomes large. We write that the limit of a sequence a_n is L in the following way:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty\,.$$

For instance

$$\lim_{n\to\infty} \frac{1}{n} = 0\,,$$

$$\lim_{n\to\infty} \frac{n+1}{n} = 1\,,$$

etc.

If a sequence has a (finite) limit then it is said to be *convergent*, otherwise it is *divergent*.

If the sequence becomes arbitrarily large then we write

$$\lim_{n\to\infty} a_n = \infty\,.$$

For instance

$$\lim_{n\to\infty} n^2 = \infty\,.$$

4.1.2. Theorem. Let f be a function defined in $[1,\infty]$. If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ for integer $n \geq 1$ then $\lim_{n\to\infty} a_n = L$ (i.e., we can replace the limit of a sequence with that of a function.)

Example: Find $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n}$.

Answer: According to the theorem that limit equals $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$, where x represents a real (rather than integer) variable. But now we can use L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{1/x}{1} = 0\,,$$

hence

$$\boxed{\lim_{n\to\infty} \frac{\ln n}{n} = 0}\,.$$

Example: Find $\displaystyle\lim_{n\to\infty} r^n$ $(r > 0)$.

Answer: This limit is the same as that of the exponential function r^x, hence

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } 0 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}}$$

4.1.3. Operations with Limits. If $a_n \to a$ and $b_n \to b$ then:

$(a_n + b_n) \to a + b$.

$(a_n - b_n) \to a - b$.

$ca_n \to ca$ for any constant c.

$a_n b_n \to ab$.

$\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$.

$(a_n)^p \to a^p$ if $p > 0$ and $a_n > 0$ for every n.

Example: Find $\lim_{n\to\infty} \frac{n^2+n+1}{2n^2+3}$.

Answer: We divide by n^2 on top and bottom and operate with limits inside the expression:

$$\lim_{n\to\infty} \frac{n^2+n+1}{2n^2+3} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{2+\frac{3}{n^2}} = \frac{1+0+0}{2+0} = \boxed{\frac{1}{2}}.$$

4.1.4. Squeeze Theorem. If $a_n \leq b_n \leq c_n$ for every $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Consequence: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Example: Find $\lim_{n\to\infty} \frac{\cos n}{n}$.

Answer: We have $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$, and $\frac{1}{n} \to 0$ as $n \to \infty$, hence by the squeeze theorem

$$\boxed{\lim_{n\to\infty} \frac{\cos n}{n} = 0}.$$

4.1.5. Other definitions.

4.1.5.1. *Increasing, Decreasing, Monotonic.* A sequence is *increasing* if $a_{n+1} > a_n$ for every n. It is *decreasing* if $a_{n+1} < a_n$ for every n. It is called *monotonic* if it is either increasing or decreasing.

Example: Prove that the sequence $a_n = \frac{n+1}{n}$ is decreasing.

Answer: $a_{n+1} - a_n = \dfrac{n+2}{n+1} - \dfrac{n+1}{n} = \dfrac{-1}{n(n+1)} < 0$, hence $a_{n+1} < a_n$ for all positive n.

4.1.5.2. *Bounded.* A sequence is *bounded above* if there is a number M such that $a_n \leq M$ for all n. It is *bounded below* if there is a number m such that $m \leq a_n$ for all n. It is called just *bounded* if it is bounded above and below.

Example: Prove that the sequence $a_n = \dfrac{n+1}{n}$ is bounded.

Answer: It is in fact bounded below because all its terms are positive: $a_n > 0$. To prove that it is bounded above note that

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \leq 2\,.$$

since $1/n \leq 1$ for all positive integer n.

4.1.6. Monotonic Sequence Theorem. Every bounded monotonic sequence is convergent.

For instance, we proved that $a_n = \dfrac{n+1}{n}$ is bounded and monotonic, so it must be convergent (in fact $\frac{n+1}{n} \to 1$ as $n \to \infty$).

Next example shows that sometimes in order to find a limit you may need to make sure that the limits exists first.

Example: Prove that the following sequence has a limit. Find it:

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$$

Answer: The sequence can be defined recursively as $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$ for $n \geq 1$. First we will prove by induction that $0 < a_n < 2$, so the sequence is bounded.

We start (base of induction) by noticing that $0 < a_1 = \sqrt{2} < 2$. Next the induction step. Assume (induction hypothesis) that for a given value of n it is true that $0 < a_n < 2$. From here we must prove that the same is true for the next value of n, i.e. that $0 < a_{n+1} < 2$. In fact $(a_{n+1})^2 = 2 + (a_n) < 2 + 2 = 4$, hence $0 < a_{n+1} < \sqrt{4} = 2$, q.e.d. So by the induction principle all terms of the sequence verify that $0 < a_n < 2$.

Now we prove that a_n is increasing:

$$(a_{n+1})^2 = 2 + a_n > a_n + a_n = 2a_n > a_n \cdot a_n = (a_n)^2 \,,$$

hence $a_{n+1} > a_n$.

Finally, since the given sequence is bounded and increasing, by the monotonic sequence theorem it has a limit L. We can find it by taking limits in the recursive relation:

$$a_{n+1} = \sqrt{2 + a_n}\,.$$

Since $a_n \to L$ and $a_{n+1} \to L$ we have:

$$L = \sqrt{2 + L} \quad \Rightarrow \quad L^2 = 2 + L \quad \Rightarrow L^2 - L - 2 = 0\,.$$

That equation has two solutions, -1 and 2, but since the sequence is positive the limit cannot be negative, hence $L = 2$.

Note that the trick works only when we know for sure that the limit exists. For instance if we try to use the same trick with the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \dots$ $(f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2})$, calling L the "limit" we get from the recursive relation that $L = L + L$, hence $L = 0$, so we "deduce" $\lim_{n\to\infty} f_n = 0$. But this is wrong, in fact the Fibonacci sequence is divergent.

4.2. Series

A *series* is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

In order to define the value of this sum we start be defining its sequence of *partial sums*

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \,.$$

Then, if $\lim_{n\to\infty} s_n = s$ exists the series is called *convergent* and its sum is that limit:

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty} s_n \,.$$

Otherwise the series is called *divergent*.

For instance, consider the following series:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \,.$$

Its partial sums are:

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \,.$$

Hence its sum is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 + 0 = 1 \,.$$

4.2.1. Geometric Series. A series verifying $a_{n+1} = r a_n$, where r is a constant, is called *geometric series*. If the first term is $a \neq 0$ then the series is

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n \,.$$

The partial sums are now:

$$s_n = \sum_{i=0}^{n} ar^i \,.$$

The nth partial sum can be found in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n \\ rs_n &= \qquad ar + ar^2 + \cdots + ar^n + ar^{n+1} \end{aligned}$$

hence

$$s_n - rs_n = a + \ 0\ +\ 0\ + \cdots +\ 0\ - ar^{n+1}\,,$$

so:

$$s_n = \frac{a(1-r^{n+1})}{1-r}\,.$$

If $|r| < 1$ we can rewrite the result like this:

$$s_n = \frac{a}{1-r} - \frac{a}{1-r} r^{n+1}\,,$$

and then get the limit as $n \to \infty$:

$$s = \lim_{n\to\infty} s_n = \frac{a}{1-r} - \frac{a}{1-r} \underbrace{\lim_{n\to\infty} r^{n+1}}_{\substack{\downarrow \\ 0}} = \frac{a}{1-r}$$

So for $|r| < 1$ the series is convergent and

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}\,.$$

For $|r| \geq 1$ the series is divergent.

4.2.2. Telescopic Series. A telescopic series is a series whose terms can be rewritten so that most of them cancel out.

Example: Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Answer: Note that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}\,. \end{aligned}$$

Hence, the sum of the series is

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = \boxed{1}.$$

4.2.3. Theorem. If the series $\sum_{n=0}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Proof: If the series is convergent then the sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$ have a limit s. On the other hand $a_n = s_n - s_{n-1}$, so taking limits we get $\lim_{n\to\infty} a_n = s - s = 0$.

The converse is not true in general. The harmonic series provides a counterexample.

4.2.4. The Harmonic Series. The following series is called *harmonic series*:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The main fact about it is that it is *divergent*. In order to prove it we find

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{2}{2} \\
s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}
\end{aligned}$$

etc., so in general $s_{2^n} > 1 + \frac{n}{2}$, hence the sequence of partial sums grows without limit and the series diverges.

4.2.5. Test for Divergence. If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Example: Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Answer: We have $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$. Since the nth term of the series does not tend to 0, the series diverges.

Example: Show that $\displaystyle\sum_{n=1}^{\infty} \sin n$ diverges.

Answer: All we need to show is that $\sin n$ does not tend to 0. If for some value of n, $\sin n \approx 0$, then $n \approx k\pi$ for some integer k, but then

$$\begin{aligned} \sin(n+1) &= \sin n \cos 1 + \cos n \sin 1 \\ &\approx \sin k\pi \cos 1 + \cos k\pi \sin 1 \\ &= 0 \pm \sin 1 \\ &= \pm 0.84 \cdots \neq 0 \end{aligned}$$

So if a term $\sin n$ is close to zero, the next term $\sin(n+1)$ will be far from zero, so it is impossible for $\sin n$ to get permanently closer and closer to 0.

4.2.6. Operations with Series. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and c is a constant then the following series are also convergent and:

(1) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(2) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(3) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

4.3. The Integral and Comparison Tests

4.3.1. The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$ is the same as that of the integral $\int_1^{\infty} f(x)\,dx$, i.e.:

(1) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The best way to see why the integral test works is to compare the area under the graph of $y = f(x)$ between 1 and ∞ to the sum of the areas of rectangles of height $f(n)$ placed along intervals $[n, n+1]$.

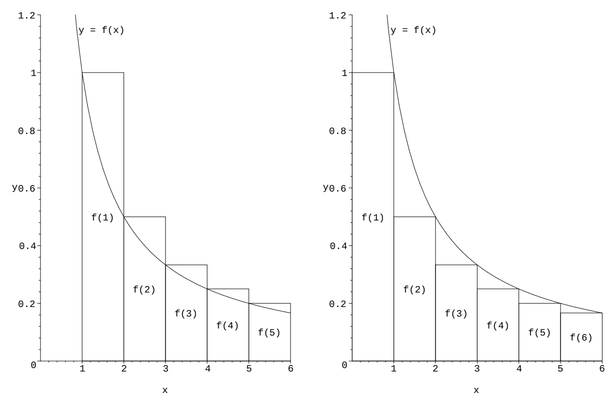


FIGURE 4.3.1

From the graph we see that the following inequality holds:

$$\int_1^{n+1} f(x)\,dx \le \sum_{i=1}^{n} a_n \le f(1) + \int_1^{n} f(x)\,dx\,.$$

The first inequality shows that if the integral diverges so does the series. The second inequality shows that if the integral converges then the same happens to the series.

Example: Use the integral test to prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.

Answer: The convergence or divergence of the harmonic series is the same as that of the following integral:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\Big[\ln x\Big]_1^t = \lim_{t\to\infty}\ln t = \infty\,,$$

so it diverges.

4.3.2. The p-series. The following series is called p-series:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\,.$$

Its behavior is the same as that of the integral $\int_1^\infty \frac{1}{x^p}\,dx$. For $p=1$ we have seen that it diverges. If $p\neq 1$ we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t = \lim_{t\to\infty}\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\,.$$

For $0<p<1$ the limit is infinite, and for $p>1$ it is zero so:

$$\boxed{\text{The } p\text{-series } \sum_{n=1}^{\infty}\frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{if } p>1 \\ \text{divergent} & \text{if } p\le 1 \end{cases}}$$

4.3.3. Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and suppose that $a_n \le b_n$ for all n. Then

(1) If $\sum b_n$ is convergent then $\sum a_n$ is convergent.
(2) If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Example: Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^2}$ converges or diverges.

Answer: We have

$$0 < \frac{\cos^2 n}{n^2} \le \frac{1}{n^2} \quad \text{for all } n\ge 1$$

and we know that the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. Hence by the comparison test, the given series also converges (incidentally, its sum is $\frac{1}{2}-\frac{\pi}{2}+\frac{\pi^2}{6} = 0.5736380465\ldots$, although we cannot prove it here).

4.3.4. The Limit Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c\,,$$

where c is a finite strictly positive number, then either both series converge or both diverge.

Example: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+4n^2}}$ converges or diverges.

Answer: We will use the limit comparison test with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/n}{1/\sqrt{1+4n^2}} &= \lim_{n\to\infty} \frac{\sqrt{1+4n^2}}{n} \\
&= \lim_{n\to\infty} \sqrt{\frac{1+4n^2}{n^2}} \\
&= \lim_{n\to\infty} \sqrt{\frac{1}{n^2}+4} \quad = \sqrt{4} = 2\,,
\end{aligned}$$

so the given series has the same behavior as the harmonic series. Since the harmonic series diverges, so does the given series.

4.3.5. Remainder Estimate for the Integral Test. The difference between the sum $s = \sum_{n=1}^{\infty} a_n$ of a convergent series and its nth partial sum $s_n = \sum_{i=1} a_i$ is the *remainder*:

$$R_n = s - s_n = \sum_{i=n+1}^{\infty} a_i\,.$$

The same graphic used to see why the integral test works allows us to estimate that remainder. Namely: If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\boxed{\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx}$$

Equivalently (adding s_n):

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx}$$

Example: Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ to the third decimal place.

Answer: We need to reduce the remainder below 0.0005, i.e., we need to find some n such that

$$\int_n^{\infty} \frac{1}{x^4}\,dx < 0.0005\,.$$

We have

$$\int_n^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_n^{\infty} = \frac{1}{3n^3}\,,$$

hence

$$\frac{1}{3n^3} < 0.0005 \quad \Rightarrow \quad n > \sqrt[3]{\frac{3}{0.0005}} = 18.17\ldots\,,$$

so we can take $n = 19$. So the sum of the 15 first terms of the given series coincides with the sum of the whole series up to the third decimal place:

$$\sum_{i=1}^{19} \frac{1}{i^4} = 1.082278338\ldots$$

From here we deduce that the actual sum s of the series is between $1.08227\ldots - 0.0005 = 1.08177\ldots$ and $1.08227\ldots + 0.0005 = 1.08277\ldots$, so we can claim $s \approx 1.082$. (The actual sum of the series is $\frac{\pi^4}{90} = 1.0823232337\ldots$.)

4.4. Other Convergence Tests

4.4.1. Alternating Series. An *alternating series* is a series whose terms are alternately positive and negative., for instance

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,.$$

4.4.1.1. *The Alternating Series Test.* If the sequence of positive terms b_n verifies

(1) b_n is decreasing.
(2) $\lim_{n\to\infty} b_n = 0$

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

Example: The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges because $1/n \to 0$. (Its sum is $\ln 2 = 0.6931471806\ldots$.)

4.4.1.2. *Alternating Series Estimation Theorem.* If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is the sum of and alternating series verifying that b_n is decreasing and $b_n \to 0$, then the remainder of the series verifies:

$$|R_n| = |s - s_n| \leq b_{n+1}\,.$$

4.4.2. Absolute Convergence. A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies convergence, i.e., if a series $\sum a_n$ is absolutely convergent, then it is convergent.

The converse is not true in general. For instance, the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent but it is not absolutely convergent.

Example: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent or divergent.

Answer: We see that the series of absolute values $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, hence the given series is absolutely convergent, therefore it is convergent (its sum turns out to be $1/4 - \pi/2 + \pi^2/6 = 0.324137741\ldots$, but the proof of this is beyond the scope of this notes).

4.4.3. The Ratio Test.

(1) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ (including $L = \infty$) then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the test is inconclusive (we do not know whether the series converges or diverges).

Example: Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$$

for absolute convergence.

Answer: We have:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \underset{n\to\infty}{\longrightarrow} e^{-1} < 1\,,$$

hence by the Ratio Test the series is absolutely convergent.

4.5. Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable of indeterminate. It can be interpreted as an infinite polynomial. The c_n's are the *coefficients* of the series. The sum of the series is a function

$$f(x) = \sum_{n=0}^{\infty} c_0 x^n$$

For instance the following series converges to the function shown for $-1 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}\,.$$

More generally given a fix number a, a *power series in* $(x-a)$, or *centered in* a, or *about* a, is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

4.5.1. Convergence of Power Series.

For a given power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ there are only three possibilities:

(1) The series converges only for $x = a$.
(2) The series converges for all x.
(3) There is a number R, called *radius of convergence*, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The *interval of convergence* is the set of values of x for which the series converges.

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}\,.$$

Answer: We use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n} = (x-3)\frac{n}{n+1} \underset{n\to\infty}{\longrightarrow} x-3\,,$$

So the power series converges if $|x-3| < 1$ and diverges if $|x-3| > 1$. Consequently, the radius of convergence is $R = 1$. On the other hand, we know that the series converges inside the interval $(2,4)$, but it remains to test the endpoints of that interval. For $x = 4$ the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}\,,$$

i.e., the harmonic series, which we know diverges. For $x = 2$ the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,,$$

i.e., the alternating harmonic series, which converges. So the interval of convergence is $[2,4)$.

4.6. Representation of Functions as Power Series

We have already seen that a power series is a particular kind of function. A slightly different matter is that sometimes a given function can be written as a power series. We already know the example

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

Replacing x with other expressions we may write other functions in the same way, for instance by replacing x with $-2x^2$ we get:

$$\frac{1}{1+2x^2} = 1 - 2x^2 + 4x^4 - 8x^6 + \cdots + (-1)^n 2^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n},$$

which converges for $|-2x^2| < 1$, i.e., $|x| < 1/\sqrt{2}$.

4.6.1. Differentiation and Integration of Power Series.

Since the sum of a power series is a function we can differentiate it and integrate it. The result is another function that can also be represented with another power series. The main related result is that the derivative or integral of a power series can be computed by *term-by-term differentiation and integration*:

4.6.1.1. *Term-By-Term Differentiation and Integration.* If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$ then the function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

(1) $$f'(x) = \sum_{n=0}^{\infty} \{c_n (x-a)^n\}' = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

(2) $$\int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the series in the above equations is R.

Example: Find a power series representation for the function

$$f(x) = \frac{1}{(1-x)^2}.$$

Answer: We have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \,,$$

hence

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=1}^{\infty} nx^{n-1} \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots = \boxed{\sum_{n=0}^{\infty}(n+1)x^n} \quad \text{(re-indexed)}\end{aligned}$$

The radius of convergence is $R = 1$.

Example: Find a power series representation for $\tan^{-1} x$.

Answer: That function is the antiderivative of $1/(1+x^2)$, hence:

$$\begin{aligned}\tan^{-1} x &= \int \frac{1}{1+x^2}\,dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx \\ &= \sum_{n=0}^{\infty}\int (-1)^n x^{2n}\,dx \\ &= C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

Since $\tan^{-1} 0 = 0$ then $C = 0$, hence

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}$$

The radius of convergence is $R = 1$.

Example: Find a power series representation for $\ln(1+x)$.

Answer: The derivative of that function is $1/(1+x)$, hence

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx \\ &= C + \sum_{n=0}^{\infty}\int (-1)^n \frac{x^{n+1}}{n+1}\,dx \\ &= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned}$$

Since $\ln 1 = 0$ then $C = 0$, so

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}}$$

The radius of convergence is $R = 1$.

4.7. Taylor and MacLaurin Series

4.7.1. Polynomial Approximations. Assume that we have a function f for which we can easily compute its value $f(a)$ at some point a, but we do not know how to find $f(x)$ at other points x close to a. For instance, we know that $\sin 0 = 0$, but what is $\sin 0.1$? One way to deal with the problem is to find an *approximate* value of $f(x)$. If we look at the graph of $f(x)$ and its tangent line at $(a, f(a))$, we see that the points of the tangent line are close to the graph, so the y-coordinates of those points are possible approximations for $f(x)$.

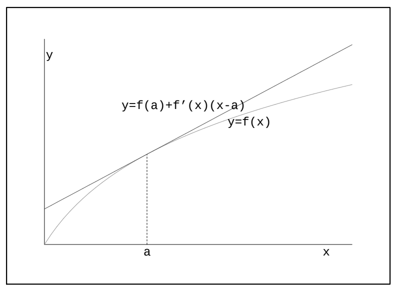


FIGURE 4.7.1. Linear approximation of $f(x)$.

The equation of the tangent line to $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a)\,,$$

hence

$$f(x) \approx f(a) + f'(a)(x - a)\,,$$

for x close to a. For instance:

$$\sin(x) \approx \sin a + \cos a\,(x - a)\,.$$

For $a = 0$ we get:

$$\sin(x) \approx \sin 0 + \cos 0 \cdot (x - 0) = x\,,$$

so $\sin(0.1) \approx 0.1$. In fact $\sin(0.1) = 0.099833416\dots$, which is close to 0.1.

The tangent line is the graph of the first degree polynomial

$$T_1(x) = f(a) + f'(a)(x - a)\,.$$

This polynomial agrees with the value and the first derivative of $f(x)$ at $x = a$:

$$\begin{aligned} T_1(a) &= f(a) \\ T_1'(a) &= f'(a) \end{aligned}$$

We can extend the idea to higher degree polynomials in the hope of obtaining closer approximations to the function. For instance, we may try a second degree polynomial of the from:

$$T_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2\,,$$

with the following conditions:

$$\begin{aligned} T_2(a) &= f(a) \\ T_2'(a) &= f'(a) \\ T_2''(a) &= f''(a) \end{aligned}$$

i.e.:

$$\begin{cases} c_0 = f(a) \\ c_1 = f'(a) \\ 2c_2 = f''(a) \end{cases}$$

After solving the system of equations obtained we get:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \end{aligned}$$

hence:

$$T_2(x) = f(a) + f'(a)x + \frac{f''(a)}{2}x^2\,.$$

In general the nth polynomial approximation of $f(x)$ at $x = a$ is an nth degree polynomial

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

verifying

$$\begin{aligned} T_n(a) &= f(a) \\ T_n'(a) &= f'(a) \\ T_n''(a) &= f''(a) \\ &\cdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

From here we get a system of $n+1$ equations with the following solution:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2!} \\ &\dots \\ c_n &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

hence:

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \,. \end{aligned}$$

That polynomial is the so called *nth-degree Taylor polynomial of $f(x)$ at $x = a$*.

Example: The third-degree Taylor polynomial of $f(x) = \sin x$ at $x = a$ is

$$T_3(x) = \sin a + \cos a \cdot (x-a)^2 - \frac{\sin a}{2}(x-a)^2 - \frac{\cos a}{3!}(x-a)^3 \,.$$

For $a = 0$ we have $\sin 0 = 0$ and $\cos 0 = 1$, hence:

$$T_3(x) = x - \frac{x^3}{6} \,.$$

So in particular

$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{6} = 0.09983333\ldots \,.$$

The actual value of $\sin 0.1$ is

$$\sin 0.1 = 0.099833416 \,,$$

which agrees with the value obtained from the Taylor polynomial up to the sixth decimal place.

4.7.2. Taylor's Inequality. The difference between the value of a function and its Taylor approximation is called *remainder*:

$$R_n(x) = f(x) - T_n(x) \,.$$

The Taylor's inequality states the following: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then the reminder satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d\,.$$

Example: Find the third degree Taylor approximation for $\sin x$ at $x = 0$, use it to find an approximate value for $\sin 0.1$ and estimate its difference from the actual value of the function.

Answer: We already found

$$T_3(x) = x - \frac{x^3}{6}\,,$$

and

$$T_3(0.1) = 0.099833333\ldots$$

Now we have $f^{(4)}(x) = \sin x$ and $|\sin x| \leq 1$, hence

$$|R_3(0.1)| =\leq \frac{1}{4!}0.1^4 = 0.0000041666\cdots < 0.0000042 = 4.2 \cdot 10^{-6}\,.$$

In fact the estimation is correct, the approximate value differs from the actual value in

$$|T_3(0.1) - \sin 0.1| = 0.000000083313\cdots < 8.34 \cdot 10^{-8}\,.$$

4.7.3. Taylor Series. If the given function has derivatives of all orders and $R_n(x) \to 0$ as $n \to \infty$, then we can write

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ &\quad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \end{aligned}$$

The infinite series to the right is called *Taylor series* of $f(x)$ at $x = a$. If $a = 0$ then the Taylor series is called *Maclaurin series.*

Example: The Taylor series of $f(x) = e^x$ at $x = 0$ is:

$$1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $|x| < d$ the remainder can be estimated taking into account that $f^{(n)}(x) = e^x$ and $|e^x| < e^d$, hence

$$|R_n(x)| < \frac{e^d}{(n+1)!}|x|^{n+1}.$$

We know that $\lim_{n\to\infty} x^n/n! = 0$, so

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$$

hence $R_n(x) \to 0$ as $n \to \infty$. Consequently we can write:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

For $x = 1$ this formula provides a way of computing number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = 2.718281828459\ldots$$

The following are Maclaurin series of some common functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = \sum_{n=0} \binom{\alpha}{n} x^n = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots$$

where $\displaystyle\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}$.

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$\tan^{-1} x = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Remark: By letting $x = 1$ in the Taylor series for $\tan^{-1} x$ we get the beautiful expression:

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}.$$

Unfortunately that series converges too slowly for being of practical use in computing π. Since the series for $\tan^{-1} x$ converges more quickly for small values of x, it is more convenient to express π as a combination of inverse tangents with small argument like the following one:

$$\boxed{\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}}.$$

That identity can be checked with plain trigonometry. Then the inverse tangents can be computed using the Maclaurin series for $\tan^{-1} x$, and from them an approximate value for π can be found.

4.7.4. Finding Limits with Taylor Series. The following example shows an application of Taylor series to the computation of limits:

Example: Find $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Answer: Replacing e^x with its Taylor series:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots) - 1 - x}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots}{x^2} \\
&= \lim_{x\to 0} \left\{ \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \dots \right\} = \boxed{\frac{1}{2}}.
\end{aligned}$$

4.8. Applications of Taylor Polynomials

4.8.1. Applications to Physics. Here we illustrate an application of Taylor polynomials to physics.

Consider the following formula from the Theory of Relativity for the total energy of an object moving at speed v:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \,,$$

where c is the speed of light and m_0 is the mass of the object at rest. Let's rewrite the formula in the following way:

$$E = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \,.$$

Now we expand the expression using the power series of the binomial function:

$$(1+x)^\alpha = \sum_{n=0} \binom{\alpha}{n} x^n = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots \,,$$

which for $\alpha = -1/2$ becomes:

$$\begin{aligned}(1+x)^{-1/2} &= 1 - \frac{1}{2}x + \binom{-\frac{1}{2}}{2} x^2 + \binom{-\frac{1}{2}}{3} x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots \,,\end{aligned}$$

hence replacing $x = -v^2/c^2$ we get the desired power series:

$$E = m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) \,.$$

If we subtract the energy at rest $m_0 c^2$ we get the kinetic energy:

$$K = E - m_0 c^2 = m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) \,.$$

For low speed all the terms except the first one are very small and can be ignored:

$$K \approx m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2} m_0 v^2 \,.$$

That is the expression for the usual (non relativistic or Newtonian) kinetic energy, so this tells us at low speed the relativistic kinetic energy is approximately equal to the non relativistic one.

4.8.2. Using Series to Solve Differential Equations. Some differential equations cannot be solved explicitly. In such cases an alternative is to represent the solution as a power series and try to determine the values of the coefficients that solve the equation. That yields a power series representation of the solution, which often is enough for getting approximations to it.

We start with an equation that we do know how to solve explicitly, so we can compare the power series obtained with the explicit solution:

$$y' = y\,.$$

This equation can be solved by separation of variables:

$$\begin{aligned}
\frac{dy}{y} &= dx \\
\int \frac{dy}{y} &= \int dx \\
\ln y &= x + C \\
y &= Ae^x \qquad (A = e^C)\,.
\end{aligned}$$

Next we solve it using power series. We start by representing the solution by a power series:

$$y = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n\,.$$

Its derivative is

$$y' = c_1 + 2c_2 x + 3c_3 x^2 \cdots = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\,.$$

Now we write the differential equation using the series:

$$c_0 + c_1 x + c_2 x^2 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + \cdots\,,$$

or

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\,.$$

In order to be equal the coefficients must be the same on both sides, so:

$$\left\{\begin{array}{lcl}
c_0 & = & c_1 \\
c_1 & = & 2c_2 \\
c_2 & = & 3c_3 \\
 & \cdots & \\
c_n & = & (n+1)c_{n+1} \\
 & \cdots &
\end{array}\right.$$

This defines a sequence of coefficients in which the first one c_0 is arbitrary, and the following ones verify the recursive relation

$$c_{n+1} = \frac{c_n}{n+1}\,.$$

So the sequence is:

$$\begin{aligned}
c_0 &= \text{(arbitrary)} \\
c_1 &= c_0 \\
c_2 &= \frac{c_1}{2} = \frac{c_0}{2} \\
c_3 &= \frac{c_2}{3} = \frac{c_0}{2\cdot 3} \\
c_4 &= \frac{c_3}{4} = \frac{c_0}{2\cdot 3\cdot 4} \\
&\cdots \\
c_n &= \frac{c_0}{n!} \\
&\cdots
\end{aligned}$$

and the solution is

$$y = c_0 + c_0 x + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \cdots = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}\,,$$

i.e., c_0 (a constant) multiplied by the Maclaurin series of e^x, so the solution is the same one we got explicitly (with $A = c_0$).

Lets look now at a more sophisticated example. Solve the differential equation

$$y'' - 2xy' + y = 0\,.$$

The idea is the same as before, we replace y with a power series, find its derivatives that appear in the equation, pose the equation with the powers series, and find a relation among the coefficients:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}
\end{aligned}$$

$$
\begin{aligned}
y'' - 2xy' + y &= \sum_{n=0}^{\infty} c_n x^n - x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}
\end{aligned}
$$

After some reindexing and grouping we get that the equation becomes:

$$
\sum_{n=0}^{\infty} \{(n+2)(n+1)c_{n+2} - (2n-1)c_n\} \, x^n = 0 \,,
$$

which implies:

$$
c_{n+2} = \frac{2n-1}{(n+1)(n+2)} \, c_n \,.
$$

The first two coefficients c_0 and c_1 are arbitrary, and the rest can be computed using that relation:

$$
\begin{aligned}
c_2 &= \frac{-1}{2} c_0 \\
c_3 &= \frac{1}{2 \cdot 3} c_1 \\
c_4 &= \frac{3}{3 \cdot 4} c_2 = -\frac{3}{4!} c_0 \\
c_5 &= \frac{5}{4 \cdot 5} c_3 = \frac{5}{5!} c_1 \\
&\cdots
\end{aligned}
$$

In general the even and odd coefficients are:

$$
\begin{aligned}
c_{2n} &= \frac{(-1) \cdot 3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{(2n)!} \, c_0 \\
c_{2n+1} &= \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} \, c_1 \,,
\end{aligned}
$$

and the solution is

$$
\begin{aligned}
y = c_0 & \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1) \cdot 3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{(2n)!} \, x^{2n} \right\} \\
& \qquad + c_1 \left\{ x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} \, x^{2n+1} \right\}
\end{aligned}
$$

APPENDIX A

Hyperbolic Functions

A.1. Hyperbolic Functions

A.1.1. Definitions. The *hyperbolic functions* are defined in the following way:

$$\begin{aligned}
\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\coth x &= \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \\
\operatorname{sech} x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \\
\operatorname{csch} x &= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}
\end{aligned}$$

The name *hyperbolic* comes from the fact that the parametric equations

$$\begin{cases} x = \cosh t \\ y = \sinh t \end{cases}$$

represent an hyperbola. Analogously, the functions $\sin x$, $\cos x$, $\tan x$, etc., are sometimes called *circular functions* because the equations

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

represent a circle.

A.1.2. Fundamental Identities. The hyperbolic functions verify some identities similar to those of the circular functions, except for some occasional sign differences:

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\coth^2 x - 1 = \operatorname{csch}^2 x$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh y$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

All of them can be verified algebraically, for instance:

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2 \\
&= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 \,.
\end{aligned}
$$

A.1.3. Derivatives of Hyperbolic Functions. The derivatives of the hyperbolic functions are easy to compute from their definitions:

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\tanh x)' = \operatorname{sech}^2 x$

$(\coth x)' = -\operatorname{csch}^2 x$

$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$

$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$

A.1.4. Integrals of Hyperbolic Functions. Reversing the derivatives found above we get:

$\displaystyle\int \sinh u \, du = \cosh u + C$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

A.1.5. Inverse Hyperbolic Functions.

The *inverse hyperbolic sine* is defined in the following way:

$$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y = \frac{1}{2}(e^y - e^{-y}) \, .$$

Solving the equation in y we get:

$$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right) \qquad \text{for all } x \, .$$

The *inverse hyperbolic cosine* is defined in the following way:

$$y = \cosh^{-1} x \quad \Leftrightarrow \quad x = \cosh y = \frac{1}{2}(e^y + e^{-y}) \quad \text{and} \quad y \geq 0 \, .$$

Solving the equation in y we get:

$$\cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right) \qquad \text{for all } x \geq 1 \, .$$

The *inverse hyperbolic tangent* is defined in the following way:

$$y = \tanh^{-1} x \quad \Leftrightarrow \quad x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} \, .$$

Solving the equation in y we get:

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right) \qquad \text{for } |x| < 1 \, .$$

Similarly we get expressions for the other inverse hyperbolic functions in terms of the natural logarithm:

$$\coth^{-1} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) \qquad \text{for } |x| > 1\,.$$

$$\operatorname{sech}^{-1} x = \ln \left(\frac{1+\sqrt{1-x^2}}{x} \right) \qquad \text{for } 0 < x \le 1\,.$$

$$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|} \right) \qquad \text{for } x \neq 0\,.$$

A.1.6. Derivatives of the Inverse Hyperbolic Functions. The derivatives of the inverse hyperbolic functions can be found from their expressions in terms of the natural logarithm, e.g.:

$$(\sinh^{-1} x)' = \left\{ \ln \left(x + \sqrt{x^2+1} \right) \right\}' = \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}\,.$$

So, we find:

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$$

$$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$$

$$(\tanh^{-1} x)' = \frac{1}{1-x^2}$$

$$(\coth^{-1} x)' = \frac{1}{1-x^2}$$

$$(\operatorname{sech}^{-1} x)' = \frac{1}{x\sqrt{1-x^2}}$$

$$(\csc^{-1} x)' = \frac{1}{|x|\sqrt{1+x^2}}$$

A.1.7. Integrals Involving Inverse Hyperbolic Functions. Reversing the derivatives found above we get the following integrals:

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u + C \qquad \text{if } |u| < 1$$

$$\int \frac{du}{1-u^2} = \coth^{-1} u + C \qquad \text{if } |u| > 1$$

$$\int \frac{du}{1-u^2} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C \qquad (|u| \neq 1)$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + C$$

$$\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + C$$

A.1.8. Taylor Series of Hyperbolic Functions. The following Taylor series involve hyperbolic functions:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad (|x| < 1)$$

APPENDIX B

Various Formulas

B.1. Summation Formulas

(1) $\displaystyle\sum_{i=1}^{n} 1 = n.$

(2) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$

(3) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$

(4) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$

APPENDIX C

Table of Integrals

Table of Integrals.

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1) \qquad \int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C \qquad \int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C \qquad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad \int \frac{du}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$$

Integrals Involving Inverse Hyperbolic Functions.

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C \qquad \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C \qquad \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$$

Reduction Formulas.

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\ \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\ \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$$

$$\int \tan^n u\,du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du\,.$$

$$\int \sec^n u\,du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} u\,du\,.$$