MATH 214-2 - Fall 2000 - Second Midterm (solutions)

SOLUTIONS

1. Differentiate the following functions

1. $f(x) = \ln(\ln x)$.
2. $f(x) = \sin(\ln 2x)$.
3. $f(x) = \sin^2(e^{-x})$.
4. $f(x) = \ln(xe^{x^2})$ (simplify first).

Solution:

1. $f'(x) = \dfrac{1}{x \ln x}$.
2. $f'(x) = \dfrac{\cos \ln 2x}{x}$.
3. $f'(x) = -2 \sin e^{-x} \cos e^{-x} e^{-x} = -e^{-x} \sin(2e^{-x})$.
4. $f(x) = \ln x + x^2 \Rightarrow f'(x) = \dfrac{1}{x} + 2x$.

2. Find the following indefinite integrals

1. $\displaystyle\int \frac{(\ln x)^2}{x}\, dx.$

2. $\displaystyle\int \cos x\, e^{\sin x}\, dx.$

Solution:

1. $u = \ln x,\ du = dx/x;$

$$\int \frac{(\ln x)^2}{x}\, dx = \int u^2\, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C\,.$$

2. $u = \sin x,\ du = \cos x;$

$$\int \cos x\, e^{\sin x}\, dx = \int e^u\, du = e^u + C = e^{\sin x} + C\,.$$

3. Find the following indefinite integrals

1. $\displaystyle\int 3^{2x}\,dx.$
2. $\displaystyle\int \frac{\log_2 x}{x}\,dx.$

Solution:

1. $u = 2x,\ du = 2\,dx;$

$$\int 3^{2x}\,dx = \frac{1}{2}\int 3^u\,du = \frac{1}{2}\frac{3^u}{\ln 3} + C = \frac{3^{2x}}{2\ln 3} + C\,.$$

2. $u = \ln x,\ du = dx/x;$

$$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx =$$
$$\frac{1}{\ln 2}\int u\,du = \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C\,.$$

4. Coopersville had a population of 25000 in 1970 and a population of 30000 in 1980. Assume that its population will continue to grow exponentially at a constant rate. What population can the Coopersville city planners expect in the year 2010?

Solution:

We may take 1970 as origin of time ($t = 0$), so 1980 corresponds to $t = 10$ and 2010 corresponds to $t = 40$. Then the population $P(t)$ at time t will be

$$P(t) = P_0 e^{kt} = 25000\, e^{kt}\,.$$

For $t = 10$ we have:

$$30000 = 25000\, e^{10k}\,,$$

hence $k = \ln(6/5)/10$. So:

$$P(40) = 25000\, e^{40k} = 25000\, e^{40 \ln(6/5)/10} =$$

$$25000\, e^{4 \ln(6/5)} = 25000 \cdot \left(\frac{6}{5}\right)^4 = 25000 \cdot \frac{1296}{625} = 51840\,.$$

5. Find the following integrals using appropriate inverse trigonometric functions:

1. $\displaystyle\int \frac{dx}{25+x^2}$.
2. $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$.

Solution:

1. $u = x/5$, $du = dx/5$;

$$\int \frac{dx}{25+x^2} = \frac{1}{25}\int \frac{dx}{1+\frac{x^2}{25}} = \frac{1}{5}\int \frac{du}{1+u^2} = \frac{1}{5}\tan^{-1} u + C = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\,.$$

2. $u = \ln x$, $du = dx/x$;

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\ln x) + C\,.$$

6. Find the following limits:

1. $L = \lim_{x\to 0} \dfrac{x - \tan^{-1} x}{x^3}$.
2. $L = \lim_{x\to\infty} \left(1 - \dfrac{1}{x^2}\right)^{x^2}$.

Solution:

1. $L = \lim_{x\to 0} \dfrac{x - \tan^{-1} x}{x^3} = \lim_{x\to 0} \dfrac{1 - \frac{1}{1+x^2}}{3x^2} =$
$$\lim_{x\to 0} \frac{x^2}{3x^2 + 3x^4} = \lim_{x\to 0} \frac{1}{3 + 3x^2} = \frac{1}{3}\,.$$
2. $\ln L = \lim_{x\to\infty} x^2 \ln\left(1 - \dfrac{1}{x^2}\right) = \lim_{x\to\infty} \dfrac{\ln\left(1 - \frac{1}{x^2}\right)}{1/x^2} = \lim_{x\to\infty} \dfrac{\frac{2/x^3}{\left(1-\frac{1}{x^2}\right)}}{-2/x^3} =$

   $\lim_{x\to\infty} \dfrac{x^2}{1 - x^2} = \lim_{x\to\infty} \dfrac{2x}{-2x} = -1.$

   Hence: $L = e^{-1}$.