MATH 214-2 - Fall 2000 - First Midterm (solutions)

SOLUTIONS

1. Find the net distance and the total distance traveled between time $t = 0$ and $t = 10$ by a particle moving at velocity $v = 4t - 12$ along a line.

Solution:

Net distance:

$$\int_0^{10} v\,dt = \int_0^{10} 4t - 12\,dt = \left[2t^2 - 12t\right]_0^{10} = 80\,.$$

Total distance:

$$\int_0^{10} |v|\,dt = \int_0^{10} |4t - 12|\,dt = \int_0^{3} -(4t - 12)\,dt + \int_3^{10} (4t - 12)\,dt$$

$$= \left[-2t^2 + 12t\right]_0^{3} + \left[2t^2 - 12t\right]_3^{10} = 18 + 98 = 116\,.$$

2. Find the volume of the solid that is generated by rotating around the x-axis the region bounded by the curves $y = 6 - x^2$ and $y = 2$.

Solution:

The intersection points of $y = 6 - x^2$ and $y = 2$ are given by the equation $6 - x^2 = 2$, i.e., $x^2 = 4$, so $x = -2$ and $x = 2$. Hence

$$V = \int_{-2}^{2} \pi(y_{\text{top}}^2 - y_{\text{bot}}^2)\,dx = \int_{-2}^{2} \pi((6 - x^2)^2 - 2^2)\,dx =$$

$$\int_{-2}^{2} \pi(x^4 - 12x^2 + 32)^2\,dx = \pi\left[\frac{x^5}{5} - 4x^3 + 32x\right]_{-2}^{2} = \frac{384\pi}{5}\,.$$

3. Use the method of cylindrical shells to find the volume of the solid generated by rotating around the y-axis the region bounded by the curves $y = x^2$ and $y = 2 - x^2$ between $x = 0$ and $x = 1$.

Solution:

$$V = \int_0^1 2\pi x(y_{\text{top}} - y_{\text{bot}})\,dx = \int_0^1 2\pi x((2 - x^2) - x^2)\,dx =$$

$$\int_0^1 2\pi(2x - 2x^3)\,dx = 2\pi\left[-\frac{x^4}{2} + x^2\right]_0^1 = 2\pi \cdot \frac{1}{2} = \pi\,.$$

4. Find the area of the surface of revolution generated by revolving the curve $y = \sqrt{x}$, $0 \le x \le 1$ around the x-axis.

Solution:

$$V = \int_{*}^{**} 2\pi y \, ds = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx =$$

$$\int_0^1 2\pi \sqrt{x} \sqrt{1 + \left((\sqrt{x})'\right)^2} \, dx = \int_0^1 2\pi \sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} \, dx =$$

$$\int_0^1 2\pi \sqrt{x + \frac{1}{4}} \, dx = 2\pi \left[\frac{2}{3}\left(x + \frac{1}{4}\right)^{3/2}\right]_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1) \, .$$

5. Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = 2xy^3(2x^2+1) \\ y(1) = 1 \end{cases}$$

Solution:

First we separate variables:

$$\frac{1}{y^3}\,dy = 2x(2x^2+1)\,dx\,.$$

Next we integrate both sides of the equation:

$$\int \frac{1}{y^3}\,dy = \int 2x(2x^2+1)\,dx + C\,,$$

i.e.:

$$-\frac{1}{2y^2} = x^4 + x^2 + C \quad\Longrightarrow\quad y^2 = \frac{1}{2(-C-x^2-x^4)}\,,$$

hence

$$y = \pm\frac{1}{\sqrt{2(-C-x^2-x^4)}}\,.$$

Now we determine the constant C by using the initial condition $x=1$, $y=1$:

$$1 = \pm\frac{1}{\sqrt{2(-C-2)}}\,.$$

In order to fulfill the initial condition the plus/minus sign must be positive, and $C = -5/2$, hence

$$y = \frac{1}{\sqrt{5-2x^2-2x^4}}\,.$$

6. Find the work done by the force $F(x) = x^2 + 1$ in moving a particle along the x-axis from $x = 0$ to $x = 6$.

Solution:

$$\int_0^6 (x^2 + 1)\, dx = \left[\frac{x^3}{3} + x\right]_0^6 = \frac{6^3}{3} + 6 = 78\,.$$