MATH 214-2 - Fall 2000 - Final Exam (solutions)

SOLUTIONS

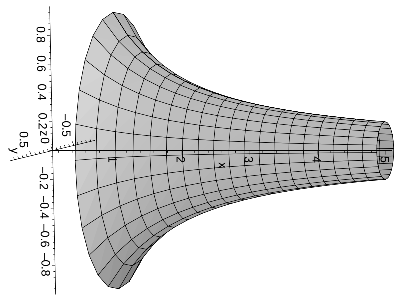
1. (Volumes By Cylindrical Shells) Use the method of cylindrical shells to find the volume of the solid generated by revolving around the y-axis the region between $y = 4x^2 - 3x$ and the x-axis from $x = 1$ to $x = 3$.

Solution:

$$V = \int_1^3 2\pi xy\,dx = \int_1^3 2\pi x(4x^2 - 3x)\,dx = 2\pi \int_1^3 (4x^3 - 3x^2)\,dx$$
$$= 2\pi \left[x^4 - x^3\right]_1^3 = 2\pi\{(81 - 27) - (1 - 1)\} = \boxed{108\,\pi}$$

2. Choose and answer **one** of the following two questions (you only need to answer **one** of these questions):

1. (Volumes by Cross-Sections and Improper Integrals) Gabriel's Horn is the surface obtained by revolving the curve $y = 1/x$, $x \geq 1$, around the x-axis. Compute the volume of revolution enclosed by Gabriel's Horn.



Gabriel's Horn

2. (Arc Length) Find the length of the curve $y = \log(\cos x)$ between $x = 0$ and $x = \pi/4$.

Solution:

1. $$V = \int_1^{\infty} \pi y^2 \, dx = \pi \int_1^{\infty} \frac{1}{x^2} \, dx = \pi \lim_{t \to \infty} \int_1^t \frac{1}{x^2} \, dx$$
$$= \pi \lim_{t \to \infty} \left[-\frac{1}{x} \right]_1^t = \pi \lim_{t \to \infty} \left(-\frac{1}{t} + y \right) = \boxed{\pi}$$

2. Since "log" may be interpreted as either decimal logarithm or natural logarithm and the wording of the problem does not make clear how it must be interpreted here, both kinds of answers will be considered correct.

   Interpreting "log" as "decimal logarithm":

$$S = \int_0^{\pi/4} ds = \int_0^{\pi/4} \sqrt{1 + (y')^2} \, dx = \boxed{\int_0^{\pi/4} \sqrt{1 + \frac{\tan^2 x}{(\ln 10)^2}} \, dx}$$

   Interpreting "log" as "natural logarithm":

$$
\begin{aligned}
S &= \int_0^{\pi/4} ds = \int_0^{\pi/4} \sqrt{1+(y')^2}\ dx = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\ dx \\
&= \int_0^{\pi/4} \sec x\ dx = \Big[\log(\sec x + \tan x)\Big]_0^{\pi/4} \\
&= \log(\sec \tfrac{\pi}{4} + \tan \tfrac{\pi}{4}) - \log(\sec 0 + \tan 0) \\
&= \log(\sqrt{2}+1) - \log 1 = \boxed{\log(\sqrt{2}+1)}
\end{aligned}
$$

3. (Separable Differential Equations) Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = \sqrt{y} \\ y(0) = 4 \end{cases}$$

Solution:

Separating variables we get:

$$\frac{dy}{\sqrt{y}} = dx\,.$$

Integrating:

$$2\sqrt{y} = x + C\,.$$

The initial condition is $y = 4$ for $x = 0$, hence:

$$2\sqrt{4} = 0 + C \quad \Rightarrow \quad C = 2\sqrt{4} = 4\,.$$

Consequently:

$$2\sqrt{y} = x + 4\,,$$

which implies:

$$\boxed{y = \left(\frac{x+4}{2}\right)^2 = \left(\frac{x}{2} + 2\right)^2}$$

4. (Exponentials, Logarithms) Simplify and differentiate the following functions:

1. $f(x) = \ln\left(\frac{\sqrt{x+1}\sqrt[3]{x^2+1}}{x^7}\right)$.

2. $g(x) = \left(e^{\sin x}\right)^{x^2}$.

Solution:

1. After simplifying we get:

$$f(x) = \frac{1}{2}\ln(x+1) + \frac{1}{3}\ln(x^2+1) - 7\ln x\,,$$

hence

$$\boxed{f'(x) = \frac{1}{2(x+1)} + \frac{2x}{3(x^2+1)} - \frac{7}{x}}$$

2. After simplifying we get:

$$g(x) = e^{x^2 \sin x}\,,$$

hence

$$g'(x) = e^{x^2 \sin x}(x^2 \sin x)' = \boxed{e^{x^2 \sin x}(2x \sin x + x^2 \cos x)}$$

5. (Natural Growth and Decay) Immediately after an accident in a nuclear power plant the level of radiation there was 16 times the safe limit. After 6 months it dropped to $8\sqrt{2}\,(\approx 11.31)$ times the safe limit. Assuming exponential decay, how long (in years) after the accident will the radiation level drop to the safe limit?

Solution:

Let $R(t)$ be the level of radiation t years after the accident. Then

$$R(t) = 16\,e^{-kt}\,.$$

After half a year the radiation level has dropped to:

$$R(\tfrac{1}{2}) = 16\,e^{-k\frac{1}{2}} = 8\sqrt{2}\,,$$

hence

$$k = -2\ln(8\sqrt{2}/16) = \ln 2\,.$$

So:

$$R(t) = 16\,e^{-t\ln 2} = 16 \cdot 2^{-t}\,.$$

So we must solve the equation

$$16 \cdot 2^{-t} = 1\,.$$

The solution is

$$t = \frac{\ln 16}{\ln 2} = \frac{\ln\,(2^4)}{\ln 2} = \frac{4\ln 2}{\ln 2} = \boxed{\text{4 years}}$$

6. (L'Hôpital's Rule) Find the following limits:

1. $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3}$
2. $\displaystyle\lim_{x\to\infty} (\sqrt{x^2+x} - x)$
3. $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$

Solution:

1. $$\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} = \lim_{x\to 0} \frac{\frac{-x^2}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{-1/3}{1+x^2} = \boxed{-\frac{1}{3}}$$

2. $$\lim_{x\to\infty} (\sqrt{x^2+x} - x) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}} - 1\right) = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}} - 1}{1/x}$$
$$= \lim_{x\to\infty} \frac{\frac{-1/x^2}{2\sqrt{1+\frac{1}{x}}}}{-1/x^2} = \lim_{x\to\infty} \frac{1}{2\sqrt{1+\frac{1}{x}}} = \boxed{\frac{1}{2}}$$

3. If $L = \lim_{x\to 0} (\cos x)^{1/x^2}$, then
$$\ln(L) = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \frac{-\tan x}{2x} = \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2},$$
hence
$$\boxed{L = e^{-1/2}}$$

7. (Integration by Parts) Find the following integral using integration by parts:

$$\int \ln x \, dx =$$

Solution:

By make $u = \ln x$, $dv = dx$, so $du = dx/x$, $v = x$:

$$\begin{aligned} \int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} &= \int u \, dv = uv - \int v \, du \\ &= x \ln x - \int x \frac{1}{x} \, dx = x \ln x - \int dx \\ &= \boxed{x \ln x - x + C} \end{aligned}$$

8. (Partial Fractions) Find the following integral by decomposing the integrand into partial fractions:

$$\int \frac{x^3 + 2x^2 + x - 1}{x^2 + x - 2}\, dx =$$

Solution:

1. Use long division and find:

$$\frac{x^3 + 2x^2 + x - 1}{x^2 + x - 2} = x + 1 + \frac{2x + 1}{x^2 + x - 2}\,.$$

2. Factor the denominator: $x^2 + x - 2 = (x - 1)(x + 2)$.

3. Decompose into partial fractions:

$$\frac{2x + 1}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}\,.$$

After multiplying by $(x - 1)(x + 2)$ we get:

$$2x + 1 = A(x + 2) + B(x - 1) = (A + B)x + (2A - B) \quad \Rightarrow$$

$$A + B = 2,\ 2A - B = 1 \quad \Rightarrow \quad A = 1,\ B = 1\,.$$

So:

$$\frac{2x + 1}{(x - 1)(x + 2)} = \frac{1}{x - 1} + \frac{1}{x + 2}\,.$$

Hence:

$$\int \frac{x^3 + x^2 + x}{x^2 + x - 2}\, dx = \int x\, dx + \int 1\, dx + \int \frac{1}{x - 1}\, dx + \int \frac{1}{x + 2}\, dx$$

$$= \boxed{\frac{x^2}{2} + x + \ln|x - 1| + \ln|x + 2| + C}$$

9. (Integrals Containing Quadratic Polynomials) Find the following integral by completing the square in the denominator:

$$\int \frac{2x}{x^2 + 6x + 10}\,dx =$$

Solution:

$$\begin{aligned}
\int \frac{2x}{x^2 + 6x + 10}\,dx &= \int \frac{2x}{(x+3)^2 + 1}\,dx \\
&= \int \frac{2u - 6}{u^2 + 1}\,du && (u = x + 3) \\
&= \int \frac{2u}{u^2 + 1}\,du + \int \frac{-6}{u^2 + 1}\,du \\
&= \ln\left(u^2 + 1\right) - 6\tan^{-1} u + C \\
&= \boxed{\ln\left(x^2 + 6x + 10\right) - 6\tan^{-1}(x + 3) + C}
\end{aligned}$$

10. (Taylor Series and Polynomials) Find the third degree Taylor polynomial of $f(x) = \tan x$ at $x = 0$. Use it to find an approximate value of $\tan(0.3)$.

Solution:

We have:

$$\begin{aligned} f(x) &= \tan x & f(0) &= 0 \\ f'(x) &= \sec^2 x & f'(0) &= 1 \\ f''(x) &= 2\sec^2 x \tan x & f''(0) &= 0 \\ f'''(x) &= 4\sec^2 x \tan^2 x + 2\sec^4 x & f'''(0) &= 2 \end{aligned}$$

Hence:

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = \boxed{x + \frac{x^3}{3}}$$

So:

$$\tan(0.3) \approx 0.3 + \frac{0.3^3}{3} = \boxed{0.309}$$

11. (Extra Credit: Hyperbolic Functions) Find the following integral using hyperbolic functions:

$$\int \frac{dx}{\sqrt{1-e^{2x}}} =$$

Solution:

Making the substitution $u = e^x$:

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-e^{2x}}} &= \int \frac{du}{u\sqrt{1-u^2}} \qquad\qquad (u = e^x) \\
&= -\operatorname{sech}^{-1}|u| + C \\
&= \boxed{-\operatorname{sech}^{-1}(e^x) + C}
\end{aligned}$$

Table of Integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1) \qquad \int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C \qquad \int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C \qquad \int \sec u\,du = \log|\sec u + \tan u| + C$$

$$\int \csc u\,du = \log|\csc u - \cot u| + C \qquad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad \int \frac{du}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$$

Integrals Involving Inverse Hyperbolic Functions

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C \qquad \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C \qquad \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$$

Reduction Formulas

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\ \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\ \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$$

$$\int \tan^n u\,du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du\,.$$

$$\int \sec^n u\,du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} u\,du\,.$$