CS 310 (sec 20) - Winter 2004 - Midterm Exam (solutions)

SOLUTIONS

1. (Logic)

(a) Prove the following logical equivalence using a truth table:

$$p \to (q \to r) \;\equiv\; (p \wedge q) \to r\,.$$

(b) Consider the following statement:

$$\exists x \exists y \exists z [(x \neq y) \wedge (x \neq z) \wedge (y \neq z) \wedge \forall t \{(t = x) \vee (t = y) \vee (t = z)\}]$$

Find a universe of discourse in which that statement is true.

Solution:

(a)

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

They have the same truth values

(b) Any universe with exactly three elements will do, e.g.: $\mathcal{U} = \{0, 1, 2\}$

2. (Proofs.) Use mathematical induction to prove the following statement for $n \geq 1$:

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}\,.$$

Solution:

1. *Basis Step*: For $n = 1$ we have

$$\frac{1}{2!} = 1 - \frac{1}{2!}\,,$$

which is obviously true.

2. *Inductive Step*: Assume that the statement is true up to some value of n. We must prove that it is also true for $n + 1$. So:

$$\begin{aligned}
\underbrace{\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!}}_{\substack{1-\frac{1}{(n+1)!} \\ \text{(induction hypothesis)}}} + \frac{n+1}{(n+2)!} &= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \\
&= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!} \\
&= 1 - \frac{1}{(n+2)!}\,,
\end{aligned}$$

which proves the statement for $n + 1$.

Hence the statement is true for every $n \geq 1$.

3. (Relations)

(a) Prove that the following is a equivalence relation on $\mathbb{Z}^+$:

$$x \,\mathcal{R}\, y \;\equiv\; \exists n \in \mathbb{Z},\; y = 2^n x \qquad \text{for every } x, y \in \mathbb{Z}^+ .$$

Describe the equivalence classes.

(b) What kind of relation do we get if we replace $\mathbb{Z}$ with $\mathbb{N} = \{0, 1, 2, \dots\}$ in the definition?:

$$x \,\mathcal{S}\, y \;\equiv\; \exists n \in \mathbb{N},\; y = 2^n x \qquad \text{for every } x, y \in \mathbb{Z}^+ .$$

Solution:

(a) The relation is

- Reflexive: $x = 2^0 x$, hence $x \,\mathcal{R}\, x$.
- Symmetric: $x \,\mathcal{R}\, y$ means $y = 2^n x$, so $x = 2^{-n} y$, which implies $y \,\mathcal{R}\, x$.
- Transitive: $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, z$ implies $y = 2^n x$ and $z = 2^m y$ for some integers n, m, so $z = 2^m 2^n x = 2^{m+n} x$, which implies $x \,\mathcal{R}\, z$.

The equivalence classes are of the form $[x] = \{2^n x \mid n = 0, 1, 2 \dots\}$, for x an odd positive integer.

(b) With the new definition the relation is still reflexive and transitive, but now it is antisymmetric: $x \,\mathcal{S}\, y$ and $y \,\mathcal{S}\, z$ implies $y = 2^n x$ and $x = 2^m y$ for some natural numbers n, m, so $x = 2^{m+n} x$, which implies $m+n = 0$. Since $m, n \geq 0$ that implies that $m = n = 0$, hence $y = 2^0 x = x$.

Consequently with the new definition the relation is a *Partial Order*.

4. (Probability) We have three boxes, one A with two red balls, another one B with one red ball and one blue ball, and a third one C with two blue balls. We pick one of the boxes with probabilities $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/6$, and take a ball from it.

1. What is the probability that the ball is red?
2. If the ball turns out to be red, what is the probability that the box picked was A?

Solution:

1. Calling $R =$ the ball taken is red, the total probability of taking a red ball is :

$$\begin{aligned} P(R) &= P(R|A)\,P(A) + P(R|B)\,P(B) + P(R|C)\,P(C) \\ &= 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{6} \\ &= \boxed{\frac{2}{3}}. \end{aligned}$$

2. Using Bayes theorem:

$$P(A) = \frac{P(R|A)\,P(A)}{P(R)} = \frac{1 \times \frac{1}{2}}{\frac{2}{3}} = \boxed{\frac{3}{4}}.$$

5. (Modular Arithmetic)

(a) Use the Euclidean algorithm to find 7^{-1} in $\mathbb{Z}_{171}$.

(b) Solve the following equation in $\mathbb{Z}_{171}$:

$$7(x+2) = 6\,.$$

Solution:

(a) In order to find $7^{-1} \pmod{171}$ we need to solve the following Diophantine equation:

$$7u + 171v = 1\,.$$

We arrange the computations as follows:

$$\begin{array}{lrcll}
\text{(divide 171 by 7)} & 171 & = & 24\cdot 7+3 & \rightarrow\ r=3 \\
\text{(divide 7 by 3)} & 7 & = & 2\cdot 3+1 & \rightarrow\ r'=1 \\
\text{(divide 3 by 1)} & 3 & = & 3\cdot 1+0 & \rightarrow\ r''=0\,,
\end{array}$$

hence $\gcd(171,7) = 1$ and the equation has a solution that can be obtained by working backward:

$$1 = 7 - 2\cdot 3 = 7 - 2\cdot(171 - 24\cdot 7) = 49\cdot 7 - 2\cdot 171\,,$$

hence $49 \cdot 7 = 1 \pmod{171}$, and $\boxed{7^{-1} = 49 \pmod{171}}$.

(b) The following operations are modulo 171:

$x + 2 = 7^{-1} \cdot 6 = 49 \cdot 6 = 294 = 123 \pmod{171}$,

$x = 123 - 2 = 121 \pmod{171}$.

Hence: $\boxed{x = 121 \pmod{171}}$.

6. (Recurrences.) Solve the following recurrence:

$$x_n = -(2x_{n-1} + x_{n-2})$$

with the initial conditions: $x_0 = 0$, $x_1 = -1$.

Solution:

The characteristic equation is:

$$r^2 + 2r + 1 = 0\,,$$

with a double root $r = -1$. So the general solution is

$$x_n = A \cdot (-1)^n + B \cdot n(-1)^n\,.$$

Now we determine A and B from the initial conditions:

$$\begin{cases} A \qquad\;\; = 0 & (n = 0) \\ -A - B = -1 & (n = 1) \end{cases}$$

The solution is $A = 0$, $B = 1$, hence:

$$\boxed{x_n = n(-1)^n}\,.$$