CS 310 (sec 20) - Winter 2004 - Final Exam (solutions)

SOLUTIONS

1. (Logic) Determine the truth value of each of the following statements:

S1: $\exists x \exists y \exists z\,(x = y + z)$

S2: $\exists x \exists y \forall z\,(x = y + z)$

S3: $\exists x \forall y \exists z\,(x = y + z)$

S4: $\exists x \forall y \forall z\,(x = y + z)$

S5: $\forall x \exists y \exists z\,(x = y + z)$

S6: $\forall x \exists y \forall z\,(x = y + z)$

S7: $\forall x \forall y \exists z\,(x = y + z)$

S8: $\forall x \forall y \forall z\,(x = y + z)$

in the universe of discourse indicated by the header of each column of the following table (write your answers in the table):

Solution:

	$\{0, 1, 2\}$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$
S1	T	T	T	T
S2	F	F	F	F
S3	T	F	T	T
S4	F	F	F	F
S5	T	T	T	T
S6	F	F	F	F
S7	F	F	T	T
S8	F	F	F	F

2. (Functions) Let $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ the following functions:

$$f(x) = (x, x, x)$$
$$g(x, y, z) = x + y + z$$

1. Find $g \circ f$.
2. Find $f \circ g$.
3. Determine if $g \circ f$ is one-to-one, onto or bijective—and in the latter case, find its inverse.
4. Same question for $f \circ g$.

Solution:

1. $(g \circ f)(x) = g(f(x)) = g(x, x, x) = x + x + x = 3x$.
2. $(f \circ g)(x, y, z) = f(g(x, y, z)) = f(x+y+z) = (x+y+z, x+y+z, x+y+z)$.
3. $g \circ f$ is bijective and its inverse is $(g \circ f)^{-1}(x) = x/3$.
4. $f \circ g$ is not one-to-one because for instance $(f \circ g)(0, 0, 0) = (0, 0, 0)$ and $(f \circ g)(1, -1, 0) = (0, 0, 0)$, so $(0, 0, 0)$ and $(1, -1, 0)$ are two different elements with the same image. It is not onto either because the image contains only elements of the form (t, t, t), so for instance $(0, 1, 2)$ is not in the image.

3. (Algorithms) Consider the following algorithm:

```
1: procedure proc(n)
2:   if n = 0 then
3:     return(1)
4:   else
5:     return(proc(n-1) + proc(n-1))
6: end proc
```

(a) Find the output of `proc(n)` for any $n \geq 0$.

(b) Assume the complexity of this algorithm is given by the number of times the **return** commands are executed. Find its complexity in Θ notation.

(c) Replace the statement in line 5 with a different statement that yields and equivalent algorithm (same output for every $n \geq 0$) of complexity $\Theta(n)$.

Solution:

(a) `proc(n)` $= 2^n$.

(b) If $a_n =$ number of times the **return** commands are executed for a given value of n, then $a_0 = 1$ and $a_{n+1} = 2a_n + 1$ for $n > 0$. Solving this recurrence we get $a_n = 2^{n+1} - 1$, hence the complexity is $\Theta(2^n)$.

(c)
```
1: procedure proc(n)
2:   if n = 0 then
3:     return(1)
4:   else
5:     return(2*proc(n-1))
6: end proc
```

4. (Combinatorics) Find the number of integer solutions of

$$x_1 + x_2 + x_3 = 15$$

subject to the conditions:

(a) $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

(b) $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$.

(c) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$.

(d) $0 \leq x_1 \leq 6$, $x_2 \geq 0$, $x_3 \geq 0$.

Solution:

(a)
$$\binom{3+15-1}{15} = \boxed{\binom{17}{15} = 136}.$$

(b) After the change of variables $x_1 = y_1 + 1$, $x_2 = y_2 + 1$, $x_3 = y_3 + 1$, the equation becomes $y_1 + y_2 + y_3 = 12$, subject to the conditions $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$. Its number of solutions is

$$\binom{3+12-1}{12} = \boxed{\binom{14}{12} = 91}.$$

(c) After the change of variables $x_1 = y_1 + 1$, $x_2 = y_2 + 2$, $x_3 = y_3 + 3$, the equation becomes $y_1 + y_2 + y_3 = 9$, subject to the conditions $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$. Its number of solutions is

$$\binom{3+9-1}{12} = \boxed{\binom{11}{9} = 55}.$$

(d) Let S_1 be the set of solutions verifying $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, S_2 the set of solutions verifying $x_1 \geq 7$, $x_2 \geq 0$, and S_3 the set of solutions verifying $0 \leq x_1 \leq 6$, $x_2 \geq 0$, $x_3 \geq 0$. Then we have

$$|S_1| = 136\,, \qquad |S_2| = \binom{3+8-1}{8} = \binom{10}{8} = 45\,,$$

and

$$|S_3| = |S_1| - |S_2| = 136 - 45 = \boxed{91}.$$

Alternatively, we can write the equation $x_2 + x_3 = 15 - x_1$, so the number of solutions is

$$\begin{aligned}\sum_{x_1=0}^{6} \binom{2+15-x_1-1}{15-x_1} &= \sum_{x_1=0}^{6} (16 - x_1) \\ &= 16+15+14+13+12+11+10 \\ &= \boxed{91}.\end{aligned}$$

5. (Graphs) In each of the following cases draw a connected simple planar graph with the given characteristics, or prove that none exists:

(a) 4 vertices all of degree 3, 4 regions.

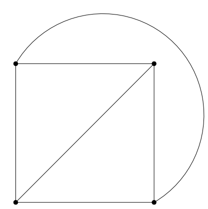
(b) 4 vertices, 6 edges, 5 regions.

(c) 4 vertices all of degree 4.

(d) 6 vertices all of degree 3, 5 regions.

Solution:

(a) The following graph fulfills the requirements:

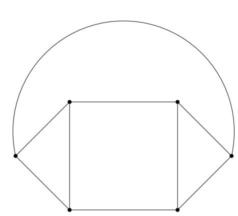


(b) No such graph exists because we have $v - e + f = 4 - 6 + 5 = 3$, contradicting Euler's formula.

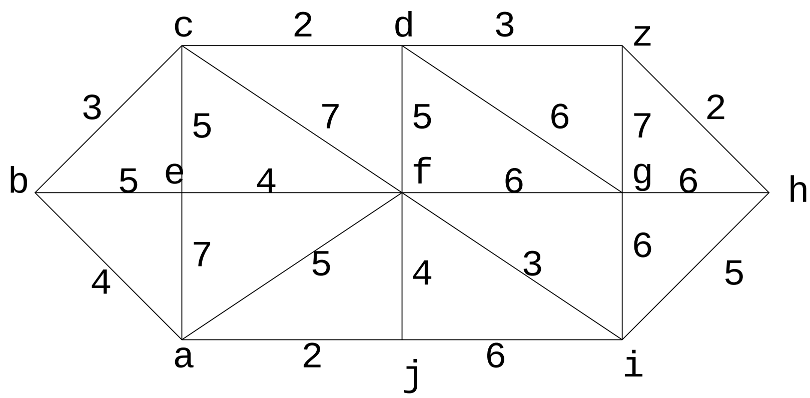
(c) No such graph exists. There are various ways to prove it:

- Such graph would have $4 \cdot 4/2 = 8$ edges, contradicting the inequality $e \leq 3v - 6$.
- In a simple graph (no parallel edges or loops) with n vertices each vertex has degree at most $n-1$ (at most one edge connecting that vertex to each of the other $n-1$ vertices).

(d) The following graph fulfills the requirements:



6. (Shortest Path) Use Dijkstra's algorithm to find the length of a shortest path, and a shortest path, from a to z in the following weighted graph:



Show also the final L-values of all vertices of the graph.

Solution:

The shortest path is $abcdz$. Its length is 12.

$L(a) = 0,\ L(b) = 4,\ L(c) = 7,\ L(d) = 9,\ L(e) = 7,\ L(f) = 5,\ L(g) = 11,$
$L(h) = 13,\ L(i) = 8,\ L(j) = 2,\ L(z) = 12.$

7. (Binary Trees) Given the following string: *abcde*

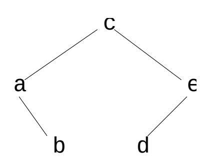
(a) Find all full binary trees of which it can be the *inorder* transversal.

(b) Find all full binary trees of which it can be the *preorder* transversal.

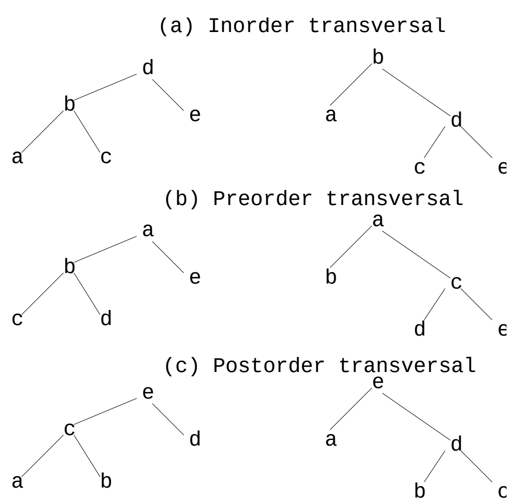
(c) Find all full binary trees of which it can be the *postorder* transversal.

Solution:

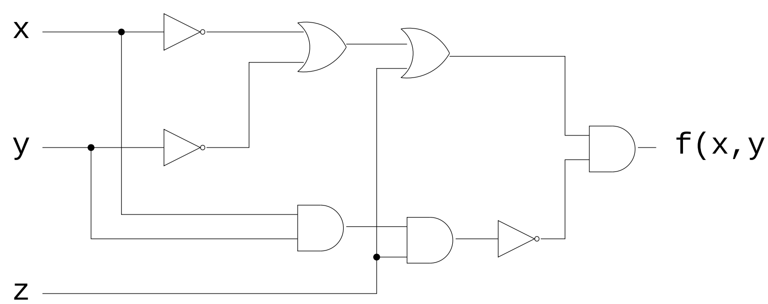
Recall that a *full binary tree* is a binary tree in which each each vertex has either two children or zero children—this reduces the number of possibilities, for instance the following is not a valid answer to part (a) because the tree is not full binary:



In a full binary tree with 5 vertices 2 of them must be interior and the other 3 are leaves, so there are two possibilities depending on whether the interior vertex different from the root is a left child of the root or a right child of the root. For each of those possibilities we get a possible answer by labeling the vertices appropriately:

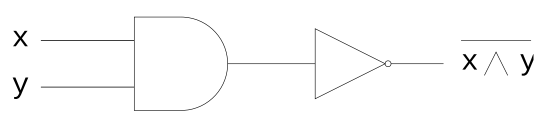


8. (Combinatorial Circuits and Boolean Algebras) Write the output $f(x, y, z)$ of the following combinatorial circuit as a Boolean expression involving x, y and z. Simplify that Boolean expression. Design an equivalent simpler circuit based on the simplified expression using the minimum possible number of gates.



Solution:

$$\begin{aligned} f(x,y,z) &= (\overline{x} \vee \overline{y} \vee z) \wedge \overline{(x \wedge y \wedge z)} \\ &= (\overline{(x \wedge y)} \vee z) \wedge (\overline{(x \wedge y)} \vee \overline{z}) \\ &= \overline{x \wedge y}\,. \end{aligned}$$



9. (Automata)

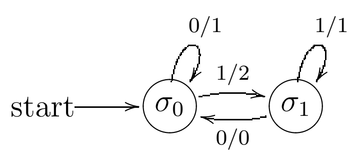
(a) Design (draw the transition diagram of) a finite-state machine that inputs any string of 0's and 1's and outputs the difference between the current and the previous symbol, plus 1 (for the first symbol assume that the "previous" symbol is 0.) For instance input "0011101" would produce output "1121102".

(b) Design a finite-state automaton that accepts the language

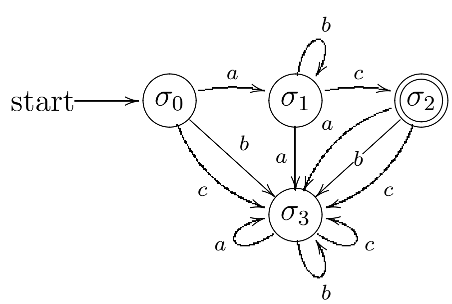
$$L = \{ab^n c \mid n = 0, 1, 2, \dots\}.$$

Solution:

(a)



(b)



10. (Languages) Let G be the grammar with terminal symbols $\{a, b\}$, non terminal symbols $\{\sigma, S\}$, productions:

$$\sigma \to \sigma b\,, \quad \sigma \to Sb\,, \quad \sigma \to aS\,, \quad S \to a$$

and starting symbol σ.

Prove that the language $L = L(G)$ associated to G is regular by finding an equivalent grammar for L that is regular.

Solution:

The language associated to G is can be described with the regular expression $a(a + b)b^*$, i.e., one a followed by a or b, followed by any number of b's.

The following is an equivalent regular grammar for L: terminal symbols $\{a, b\}$, non terminal symbols $\{\sigma, A, B\}$, productions:

$$\sigma \to aA\,, \quad A \to aB\,, \quad A \to bB\,, \quad B \to bB\,, \quad B \to \lambda\,,$$

and starting symbol σ.