CS 310 (sec 20) - Winter 2003 - Midterm Exam (solutions)

SOLUTIONS

1. (Proofs.) Use mathematical induction to prove the following statement for $n \geq 1$:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}\,.$$

Solution:

1. *Basis Step*: For $n = 1$ we have

$$\frac{1}{1 \cdot 2} = 1 - \frac{1}{1 \cdot 2}\,,$$

which is obviously true.

2. *Inductive Step*: Assume that the statement is true up to some value of n. We must prove that it is also true for $n+1$. So:

$$\underbrace{\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}}_{\substack{1-\frac{1}{n+1} \\ \text{(induction hypothesis)}}} + \frac{1}{(n+1)(n+2)}$$

$$= 1 - \frac{1}{n+1} + \underbrace{\frac{1}{(n+1)(n+2)}}_{\frac{1}{n+1}-\frac{1}{n+2}}$$

$$= 1 - \frac{1}{n+2}\,,$$

which proves the statement for $n+1$.

Hence the statement is true for every $n \geq 1$.

2. (Relations.) For each of the following relations defined on $\mathbb{R}$ determine whether it is reflexive, symmetric, antisymmetric, transitive:

1. $x\mathcal{R}y$ if $x = 2y$.
2. $x\mathcal{R}y$ if $x = ky$ for some integer $k \in \mathbb{Z}$.
3. $x\mathcal{R}y$ if $x - y \in \mathbb{Q}$.
4. $x\mathcal{R}y$ if $x - y \in \mathbb{Z}^+$.

Solution:

1. Antisymmetric.
2. Reflexive, transitive.
3. Reflexive, symmetric, transitive.
4. Antisymmetric, transitive.

Explanation:

1. In general $x \neq 2x$ (unless $x = 0$), so not reflexive. $x = 2y$ does not imply $y = 2x$, so not symmetric. $x = 2y$ and $y = 2x$ implies $x = 4x$, which implies $x = 0$ and $y = 0$, so $x = y$, hence it is antisymmetric. $x = 2y$ and $y = 2x$ implies $x = 4y$, which does not imply $x = 2y$, so not transitive.
2. $x = 1 \cdot x$, so it is reflexive. $x = ky$ does not imply $y = k'x$ (e.g. $6 = 2 \cdot 3$, but $3 \neq k \cdot 6$ for $k \in \mathbb{Z}$), so not symmetric. $x = (-1) \cdot (-x)$ and $-x = (-1) \cdot x$, but $x \neq -x$ (unless $x = 0$), so not antisymmetric. $x = ky$ and $y = k'z$ implies $x = kk'z$ and $kk' \in \mathbb{Z}$, so it is transitive.
3. $x - x = 0 \in \mathbb{Q}$, so it is reflexive. $x - y = r \in \mathbb{Q}$ implies $y - x = -r \in \mathbb{Q}$, so it is symmetric. $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$ does not imply $x = y$ (e.g. consider $x = 1$, $y = 0$, so $1 - 0 = 1 \in \mathbb{Q}$ and $0 - 1 = -1 \in \mathbb{Q}$, but $0 \neq 1$, so not antisymmetric. $x - y = r \in \mathbb{Q}$ and $y - z = r' \in \mathbb{Q}$ implies $x - z = r + r' \in \mathbb{Q}$, so it is transitive.
4. $x - x = 0 \notin \mathbb{Z}^+$, so not reflexive. $x - y = r \in \mathbb{Z}^+$ does not imply $y - x = -r \in \mathbb{Z}^+$, so not symmetric. $x - y \in \mathbb{Z}^+$ and $y - z \in \mathbb{Z}^+$ is always false ($x - y$ and $y - x$ cannot be both positive), so the implication $(x - y \in \mathbb{Z}^+) \wedge (y - z \in \mathbb{Z}^+) \implies x = y$ is vacuously true, hence the relation is antisymmetric. $x - y = k \in \mathbb{Z}^+$ and $y - z = k' \in \mathbb{Z}^+$ implies $x - z = k + k' \in \mathbb{Z}^+$, so it is transitive.

3. (Functions.) Let $f, g : \mathbb{R} \to \mathbb{R}$ the functions $f(x) = 2x + 2$, $g(x) = x/2$. Find $f \circ g$, $g \circ f$, f^{-1}, g^{-1}, $(f \circ g)^{-1}$, $(g \circ f)^{-1}$, $f^{-1} \circ g^{-1}$ and $g^{-1} \circ f^{-1}$.

Solution:

$(f \circ g)(x) = f(g(x)) = 2g(x) + 2 = x + 2.$

$(g \circ f)(x) = g(f(x)) = f(x)/2 = x + 1.$

$f^{-1}(x) = \dfrac{x-2}{2}.$

$g^{-1}(x) = 2x.$

$(f \circ g)^{-1}(x) = x - 2.$

$(g \circ f)^{-1}(x) = x - 1.$

$(f^{-1} \circ g^{-1})(x) = x - 1.$

$(g^{-1} \circ f^{-1})(x) = x - 2.$

4. (Modular Arithmetic.) Use the Euclidean algorithm for finding a number x between 0 and 62 such that $10x \equiv 1 \pmod{63}$.

Solution:

We use the Euclidean algorithm for finding $\gcd(63, 10) = 1$ and solving simultaneously the Diophantine equation $10x + 63y = 1$.

$$\begin{array}{rcl} 63 & = & 6 \cdot 10 + 3 \quad \rightarrow \quad 3 = 63 - 6 \cdot 10 \\ 10 & = & 3 \cdot 3 + 1 \quad \rightarrow \quad 1 = 10 - 3 \cdot 3 \end{array}$$

Hence: $1 = 10 - 3 \cdot 3 = 10 - 3 \cdot (63 - 6 \cdot 10) = 3 \cdot 63 + 19 \cdot 10$; so: $19 \equiv 10 \pmod{63}$. Hence the answer is $\boxed{x = 19}$.

5. (Counting.) Consider all possible strings of length 8 made of a's, b's and c's, for instance *abbcbacb*.

1. How many of them are there?
2. How many of them have 5 a's, 2 b's and 1 c?
3. How many of them have exactly 3 a's?

You do not need to compute the final answer completely.

Solution:

1. Three choices for each of 8 letters: $\boxed{3^8 = 6561}$.
2. Permutations with repeated elements: $P(8;5,2,1) = \boxed{\frac{8!}{5!2!1!} = 168}$.
3. There are 2^5 strings of a's and b's of length 5, and we permute them with 3 a's: $P(8;3,5) \cdot 2^5 = \boxed{\frac{8!}{3!5!} \cdot 2^5 = 1792}$.

6. (Recurrences.) Solve the following recurrence:

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$

with the initial conditions: $x_0 = 0$, $x_1 = 1$.

Solution:

The characteristic equation is:

$$2x^2 - x - 1 = 0\,,$$

which has the roots $x_1 = 1$, $x_2 = -1/2$. So the general solution is

$$x_n = A \cdot 1^n + B(-1/2)^n\,.$$

Now we determine A and B from the initial conditions:

$$\begin{cases} A + \phantom{\tfrac{1}{2}}B = 0 & (n = 0) \\ A - \frac{1}{2}B = 1 & (n = 1) \end{cases}$$

The solution is $A = 2/3$, $B = -2/3$, hence:

$$\boxed{x_n = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n}\,.$$