CS 310 - Winter 2001 - Midterm Exam (solutions)

SOLUTIONS

1. (Logic)

(a) Prove the following logical equivalence by using Laws of Logic (Algebra of Propositions):

$$p \to (q \to r) \;\Leftrightarrow\; (p \land q) \to r\,.$$

(Assume that '$\to$' is defined by "$p \to q \Leftrightarrow \neg p \lor q$".)

Solution:

$$p \to (q \to r) \overset{\text{(Def. of '}\to\text{')}}{\Longleftrightarrow} \neg p \lor (\neg q \lor r) \overset{\text{(Associative)}}{\Longleftrightarrow} (\neg p \lor \neg q) \lor r$$
$$\overset{\text{(DeMorgan's)}}{\Longleftrightarrow} \neg(p \land q) \lor r \overset{\text{(Def. of '}\to\text{')}}{\Longleftrightarrow} (p \land q) \to r$$

(b) Determine the truth value of each of the following statements:

S1: $\exists x \forall y \exists z\, (x = y + z)$
S2: $\forall x \forall y \exists z\, (x = y + z)$
S3: $\forall x \exists y\, [(x < y) \land \forall z\, (x < z \to y \leq z)]$

in the universe of discourse indicated by the header of each column of the following table (write your answers in the table):

Solution:

	$\{0,1,2\}$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$
S1	1	0	1	1
S2	0	0	1	1
S3	0	1	1	0

Remarks: S2 basically means that the difference $x - y$ of any two elements of $\mathcal{U}$ (universe of discourse) is also in $\mathcal{U}$—this is true in $\mathbb{Z}$ and $\mathbb{Q}$, but it is false in $\mathbb{N}$ and $\{0, 1, 2\}$. S1 means the same but just for some x—so we may take say $x = 0$ and see that the statement is true in $\mathbb{Z}$ and $\mathbb{Q}$. However it is false in $\mathbb{N}$ because $x - y \notin \mathbb{N}$ if $x < y$. On the other hand, taking $x = 2$ we see that it is true in $\{0, 1, 2\}$. Here S3 can be interpreted as "every element has an immediate successor"—true in $\mathbb{N}$ and $\mathbb{Z}$, but false in $\{0, 1, 2\}$ (2 has no successor) and $\mathbb{Q}$ (the order in $\mathbb{Q}$ is dense, i.e., between two rational numbers there is always another rational number.)"

2. (Relations) On $\mathbb{C}$ (set of complex numbers) we define the relations

$$x \,\mathcal{R}\, y \Leftrightarrow \exists n \in \mathbb{N},\; x + n = y$$

and

$$x \,\mathcal{S}\, y \Leftrightarrow \exists n \in \mathbb{Z},\; x + n = y$$

(a) Prove that $\mathcal{R}$ is a partial order.

(b) Prove that $\mathcal{S}$ is an equivalence relation.

Solution:

(a) $\mathcal{R}$ is a partial order:

- Reflexive: $x + 0 = x \Rightarrow x \,\mathcal{R}\, x$.
- Antisymmetric: We have:

  $$x \,\mathcal{R}\, y \Leftrightarrow \exists n \in \mathbb{N},\; x + n = y$$

  and

  $$y \,\mathcal{R}\, x \Leftrightarrow \exists n' \in \mathbb{N},\; y + n' = x$$

  Adding the last equations we get $n + n' = 0$, but since they must be natural numbers, we conclude $n = n' = 0$, which implies $x = y$.
- Transitive:

  $$\left.\begin{array}{l} x \,\mathcal{R}\, y \Rightarrow \exists n \in \mathbb{N},\; x + n = y \\ y \,\mathcal{R}\, z \Rightarrow \exists n' \in \mathbb{N},\; y + n' = z \end{array}\right\} \Rightarrow x + n + n' = z \Rightarrow x \,\mathcal{R}\, z\,.$$

(b) $\mathcal{S}$ is an equivalence relation:

- Reflexive: $x + 0 = x \Rightarrow x \,\mathcal{S}\, x$.
- Symmetric: $x \,\mathcal{S}\, y \Rightarrow \exists n \in \mathbb{Z},\; x + n = y \Rightarrow y + (-n) = x \Rightarrow y \,\mathcal{S}\, x$.
- Transitive:

  $$\left.\begin{array}{l} x \,\mathcal{S}\, y \Rightarrow \exists n \in \mathbb{Z},\; x + n = y \\ y \,\mathcal{S}\, z \Rightarrow \exists n' \in \mathbb{Z},\; y + n' = z \end{array}\right\} \Rightarrow x + n + n' = z \Rightarrow x \,\mathcal{S}\, z\,.$$

3. (Functions) For each one of the following functions, determine if it is one-to-one (but not onto), onto (but not one-to-one), or a one-to-one correspondence. If it is a one-to-one correspondence, find its inverse.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$.

(b) $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x + 1$.

(c) $f : \mathbb{Z} \to \mathbb{N}$, $f(x) = |x| =$ "absolute value of x".

Solution:

(a) One-to-one (but not onto): $f(x) = f(y) \Rightarrow 2x + 1 = 2y + 1 \Rightarrow x = y$, so f is one-to-one. But $f(x) = 2x+1$ takes only odd values, $f(\mathbb{Z})$ does not contain even numbers, so f is not onto.

(b) One-to-one correspondence: the inverse is $f^{-1}(x) = (x-1)/2$. In fact $f^{-1} \circ f(x) = f^{-1}(f(x)) = ((2x+1)-1)/2 = 2x/2 = x = \mathrm{id}_{\mathbb{Q}}(x)$, and $f \circ f^{-1}(x) = f(f^{-1}(x)) = 2((x-1)/2) + 1 = (x-1) + 1 = x = \mathrm{id}_{\mathbb{Q}}(x)$. Since f has an inverse, it is indeed a one-to-one correspondence.

(c) Onto but not one-to-one. If $y \in \mathbb{N}$ we must prove that there is some $x \in \mathbb{Z}$ such that $f(x) = |x| = y$. This can be accomplished by taking $x = y$, since for any $x \in \mathbb{N}$, $f(x) = |x| = x$. On the other hand it is not one-to-one, since there are different elements in $\mathbb{Z}$ with the same image, e.g., $f(1) = |1| = 1$, $f(-1) = |-1| = 1$, so $f(1) = f(-1)$.

4. (Functions and Sets) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x, y) = (2x+y, 2x-y)$, and let D be the following subset of $\mathbb{R}^2$:

$$D = \{(a, a) \mid a \in \mathbb{R}\} .$$

Find (a) $f(D)$, (b) $f^{-1}(D)$, (c) $f(D) \cap f^{-1}(D)$.

Solution:

(a) $f(D) = \{f(a, a) \mid a \in \mathbb{R}\} = \{(3a, a) \mid a \in \mathbb{R}\}$.
Another way to express the solution is $f(D) = \{(x, y) \in \mathbb{R}^2 \mid x = 3y\}$.

(b) We have: $(x, y) \in f^{-1}(D) \Leftrightarrow f(x, y) \in D \Leftrightarrow (2x + y, 2x - y) \in D \Leftrightarrow 2x + y = 2x - y \Leftrightarrow y = 0$. Hence:

$$f^{-1}(D) = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} .$$

or

$$f^{-1}(D) = \{(x, 0) \mid x \in \mathbb{R}\} .$$

(c) $(x, y) \in f(D) \cap f^{-1}(D) \Leftrightarrow x = 3y$ and $y = 0 \Leftrightarrow (x, y) = (0, 0)$, hence:

$$f(D) \cap f^{-1}(D) = \{(0, 0)\} .$$

5. (Operations) Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operation defined on the real interval $[1, \infty)$:

$$x * y = \sqrt{x^2 + y^2 - 1}\,.$$

Solution:

- Commutative property:

$$x * y = \sqrt{x^2 + y^2 - 1} = \sqrt{y^2 + x^2 - 1} = y * x\,.$$

- Associative property:

$$(x * y) * z = \sqrt{\left(\sqrt{x^2 + y^2 - 1}\right)^2 + z^2 - 1} = \sqrt{x^2 + y^2 + z^2 - 2}\,,$$
$$x * (y * z) = \sqrt{x^2 + \left(\sqrt{y^2 + z^2 - 1}\right)^2 - 1} = \sqrt{x^2 + y^2 + z^2 - 2}\,,$$

  hence $(x * y) * z = x * (y * z)$.

- Identity element. The identity element is 1:

$$1 * x = x * 1 = \sqrt{x^2 + 1 - 1} = x\,.$$

- Inverse element: Given an $x \in [1, \infty)$, its inverse x' must verify:

$$1 = x * x' = \sqrt{x^2 + x'^2 - 1}\,,$$

  hence,

$$x^2 + x'^2 = 2\,.$$

  Since $x, x' \geq 1$, necessarily $x = x' = 1$. So the only invertible element is 1.

6. (Counting) Consider the following equation:

$$x_1 + x_2 + x_3 = 15\,.$$

(a) How many non negative integer solutions does it have?

(b) How many of those solutions are strictly positive?

(c) How many non negative solutions consist of numbers divisible by three only?

(d) How many non negative solutions verify that x_1 is a multiple of 5 (including 0)?

Do not try to find the solutions, just compute their number.

Solution:

(a) $\binom{3+15-1}{15} = \binom{17}{15} = 136$.

(b) Calling $x_1 = y_1 + 1$, $x_2 = y_2 + 1$, $x_3 = y_3 + 1$, the equation becomes:

$$y_1 + y_2 + y_3 = 15 - 3 = 12\,.$$

Its non negative solutions correspond to strictly positive solutions to the original equation. Their number is $\binom{3+12-1}{12} = \binom{14}{12} = 91$.

(c) Calling $x_1 = 3z_1$, $x_2 = 3z_2$, $x_3 = 3z_3$, the equation becomes:

$$z_1 + z_2 + z_3 = 5\,.$$

Its non negative solutions correspond to non negative multiple of three solutions to the original equation. Their number is $\binom{3+5-1}{5} = \binom{7}{5} = 21$.

(d) The possible values of x_1 are 0, 5, 10 and 15, so the problem requires to count the number of solutions to the following equations:

$$\begin{aligned}
0 + x_2 + x_3 = 15 &\quad \text{or} \quad x_2 + x_3 = 15 \\
5 + x_2 + x_3 = 15 &\quad \text{or} \quad x_2 + x_3 = 10 \\
10 + x_2 + x_3 = 15 &\quad \text{or} \quad x_2 + x_3 = \ \ 5 \\
15 + x_2 + x_3 = 15 &\quad \text{or} \quad x_2 + x_3 = \ \ 0
\end{aligned}$$

The answer is $\binom{16}{15} + \binom{11}{10} + \binom{6}{5} + \binom{1}{0} = 16 + 11 + 6 + 1 = 34$.