CS 310 - Winter 2001 - Final Exam (solutions)

SOLUTIONS

1. (Logic) Prove the following logical equivalence using truth tables:

$$p \to (q \to \mathrm{F}_0) \quad \Leftrightarrow \quad q \to \neg p$$

where F_0 represents *falsehood* and its truth value is always *false*.

Solution:

The truth table is as follows:

p	q	F_0	$q \to \mathrm{F}_0$	$p \to (q \to \mathrm{F}_0)$	$\neg p$	$q \to \neg p$
0	0	0	1	1	1	1
0	1	0	0	1	1	1
1	0	0	1	1	0	1
1	1	0	0	0	0	0

We see that the columns for $p \to (q \to \mathrm{F}_0)$ and $q \to \neg p$ are identical, so they are logically equivalent.

2. (Sets) Remember that a *partition* of a set A is a collection of non-empty subsets $A_1, A_2, A_3, \ldots$ of A which are pairwise disjoint and whose union equals A. Find all possible partitions of the set $A = \{a, b, c\}$.

Solution:

There are five partitions:

1. $A_1 = \{a, b, c\}$.
2. $A_1 = \{a, b\}$, $A_2 = \{c\}$.
3. $A_1 = \{a, c\}$, $A_2 = \{b\}$.
4. $A_1 = \{b, c\}$, $A_2 = \{a\}$.
5. $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$.

3. (Relations) On $\mathbb{Z}^+$ we define the relation $x \,\mathcal{R}\, y \Leftrightarrow \exists k \in \mathbb{N}\,, y = 2^k x$.

1. Prove that $\mathcal{R}$ is a partial order.
2. Draw the Hesse diagram of $(S, \mathcal{R})$, where $S = \{2, 3, 4, 5, 6, 8, 10\}$.
3. Find, if they exist, the least element, the greatest element, the minimal elements and the maximal elements of $(S, \mathcal{R})$.

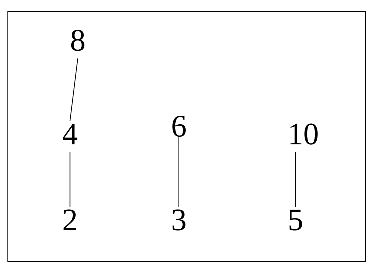
Solution:

1. (a) Reflexive property: $x = 2^0 x \Rightarrow x \,\mathcal{R}\, x$.

   (b) Antisymmetric property: $x \,\mathcal{R}\, y \Rightarrow y = 2^k x$ for some $k \in \mathbb{N}$; $y \,\mathcal{R}\, x \Rightarrow x = 2^{k'} y$ for some $k' \in \mathbb{N}$; hence $xy = 2^{k+k'} xy \Rightarrow 2^{k+k'} = 1 \Rightarrow k + k' = 0$. Since k and k' are natural numbers the only solution is $k = k' = 0$, hence $y = 2^k x = 2^0 x = x$, i.e., $x = y$.

   (c) Transitive property: $x \,\mathcal{R}\, y \Rightarrow y = 2^k x$ for some $k \in \mathbb{N}$; $y \,\mathcal{R}\, z \Rightarrow z = 2^{k'} y$ for some $k' \in \mathbb{N}$; hence $z = 2^{k'} 2^k x = 2^{k'+k} x \Rightarrow x \,\mathcal{R}\, z$.

2. Hesse diagram of S:



3. There is no least or greatest element in S.

   Minimal elements of S: $2, 3, 5$.

   Maximal elements of S: $6, 8, 10$.

4. (Functions) Let $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ the following functions:

$$f(x) = (x, 2x)$$
$$g(x, y) = x + y$$

1. Find $g \circ f$.
2. Find $f \circ g$.
3. Determine if $g \circ f$ is one-to-one, onto or a one-to-one correspondence—and in the latter case, find its inverse.
4. Same question for $f \circ g$.

Solution:

1. $(g \circ f)(x) = g(f(x)) = g(x, 2x) = x + 2x = 3x$.
2. $(f \circ g)(x, y) = f(g(x, y)) = f(x + y) = (x + y, 2x + 2y)$.
3. $g \circ f$ is a one-to-one correspondence, and its inverse is $(g \circ f)^{-1}(x) = x/3$.
4. $f \circ g$ is not one-to-one because for instance $(f \circ g)(0, 0) = (0, 0)$ and $(f \circ g)(1, -1) = (0, 0)$, so $(0, 0)$ and $(1, -1)$ are two different elements with the same image. It is not onto either because the image contains only elements of the form $(t, 2t)$, so for instance $(0, 1)$ is not in the image.

5. (Operations) Find the properties (commutative, associative, identity element, inverse element) verified by the following operation on $\mathbb{R}$:

$$x \circ y = \frac{x+y}{2}\,.$$

Solution:

1. It is commutative: $x \circ y = \dfrac{x+y}{2} = \dfrac{y+x}{2} = y \circ x$.
2. It is NOT associative. Counterexample: $(1 \circ 2) \circ 3 = \dfrac{\frac{1+2}{2}+3}{2} = \dfrac{9}{4}$; $1 \circ (2 \circ 3) = \dfrac{1+\frac{2+3}{2}}{2} = \dfrac{7}{4}$, hence $(1 \circ 2) \circ 3 \neq 1 \circ (2 \circ 3)$.
3. There is NO identity element:

   $$x \circ e = \frac{x+e}{2} = x \quad \Rightarrow \quad e = x\,,$$

   so there is no fix element e such that $x \circ e = e \circ x = x$ for every $x \in \mathbb{R}$.
4. Inverse element: Since there is no identity element, there cannot be inverse element either.

6. (Counting) Let $A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$ be the set of 26 letters of the English alphabet.

1. How many subsets does A have?
2. How many of them contain the letter a?
3. How many of them do not contain the letter a?
4. How many of them contain all five vowels a, e, i, o, u?
5. How many of them do not contain any vowel?
6. How many of them contain at least one vowel?

Solution:

1. The number of subsets of A is $|\mathcal{P}(A)| = 2^{26} = 67108864$.
2. If a subset S of A contains the letter a then it is of the form $S = \{a\} \cup S'$ where $S' \subseteq A - \{a\}$. Hence the number of subsets of A containing the letter a equals the number of subsets of $A - \{a\}$, which is $|\mathcal{P}(A - \{a\})| = 2^{25} = 33554432$.
3. The subsets of A that do not contain the letter a are the subsets of $A - \{a\}$, and their number is $|\mathcal{P}(A - \{a\})| = 2^{25} = 33554432$.
4. The subsets S of A containing all five vowels a, e, i, o, u are of the form $S = \{a, e, i, o, u\} \cup S'$, where $S' \subseteq A - \{a, e, i, o, u\}$. Their number is $|\mathcal{P}(A - \{a, e, i, o, u\})| = 2^{21} = 2097152$.
5. The subsets of A not containing any vowel are the subsets of $A - \{a, e, i, o, u\}$, and their number is $|\mathcal{P}(A - \{a, e, i, o, u\})| = 2^{21} = 2097152$.
6. The subsets containing at least one vowel are the elements of $\mathcal{P}(A)$ that are not in $\mathcal{P}(A - \{a, e, i, o, u\})$. Their number is $|\mathcal{P}(A) - \mathcal{P}(A - \{a, e, i, o, u\})| = |\mathcal{P}(A)| - |\mathcal{P}(A - \{a, e, i, o, u\})| = 2^{26} - 2^{21} = 65011712$.

7. (Recurrences) Solve the following recurrence:

$$x_n = -2x_{n-1} - x_{n-2}$$

with the initial conditions: $x_0 = 0$, $x_1 = -1$.

Solution:

The characteristic equation is:

$$x^2 + 2x + 1 = 0\,,$$

which has a double root $x = -1$. So the general solution is

$$x_n = A(-1)^n + Bn(-1)^n\,.$$

Now we determine A and B from the initial conditions:

$$\begin{cases} A \quad\;\; = \;\; 0 & (n = 0) \\ -A - B = -1 & (n = 1) \end{cases}$$

The solution is $A = 0$, $B = 1$, hence:

$$x_n = n(-1)^n\,.$$

8. (Modular Arithmetic) Find $27^{-1} \mod 128$. [Hint: solve $27x + 128y = 1$.]

Solution:

We use the Euclidean algorithm:

$$\begin{array}{rcll} 128 &=& 4\cdot 27+20 & \rightarrow\ 20 = 128 - 4\cdot 27 \\ 27 &=& 1\cdot 20+7 & \rightarrow\ 7 = 27 - 1\cdot 20 \\ 20 &=& 2\cdot 7+6 & \rightarrow\ 6 = 20 - 2\cdot 7 \\ 7 &=& 1\cdot 6+1 & \rightarrow\ 1 = 7 - 1\cdot 6 \end{array}$$

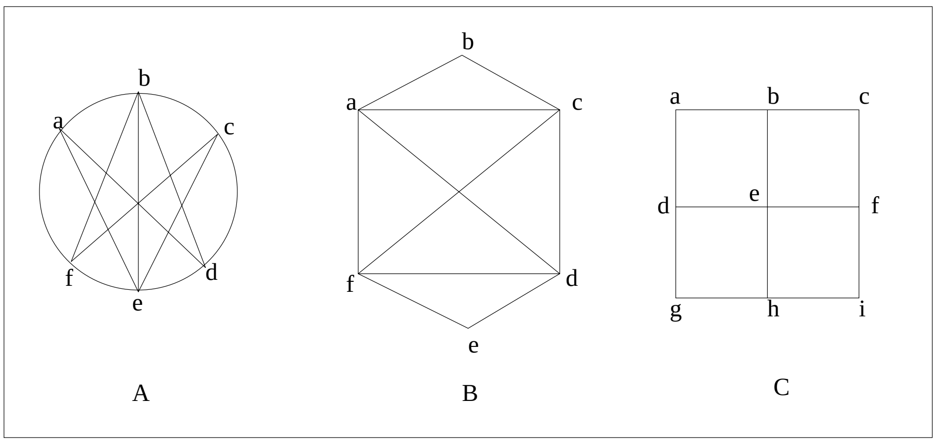
Hence: $1 = 7-1\cdot 6 = 7-1\cdot(20-2\cdot 7) = 3\cdot 7-1\cdot 20 = 3\cdot(27-1\cdot 20)-1\cdot 20 = 3\cdot 27-4\cdot 20 = 3\cdot 27-4\cdot(128-4\cdot 27) = 19\cdot 27-4\cdot 128$.

So: $19\cdot 27 = 1+4\cdot 128 = 1 \mod 128$, hence

$$27^{-1} \mod 128 = 19\,.$$

9. (Graphs) For each of the following graphs determine if:

1. It has an Euler trail that is not a circuit.
2. It has an Euler circuit.
3. It has a Hamilton path.
4. It has a Hamilton cycle.
5. It is planar.



Justify your answers (i.e., tell "why" so that it can be seen that your answer is not just a lucky guess.)

Solution:

1. Graph A:
   (a) It has an Euler trail that is not a circuit because it has two vertices with odd degree (b and e).
   (b) For the same reason it does NOT have an Euler circuit.
   (c) It has a Hamilton path, e.g.: $a\,b\,c\,d\,e\,f$.
   (d) It has a Hamilton cycle, e.g.: $a\,b\,c\,d\,e\,f\,a$.
   (e) It is not planar because it contains $K_{3,3}$.
2. Graph B:

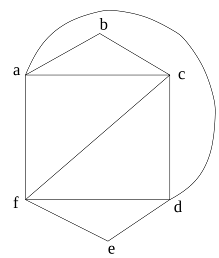
(a) It has NO Euler trail that is not a circuit because all its vertices have even degree.

(b) For the same reason it does have an Euler circuit.

(c) It has a Hamilton path, e.g.: $a\,b\,c\,d\,e\,f$.

(d) It has a Hamilton cycle, e.g.: $a\,b\,c\,d\,e\,f\,a$.

(e) It is planar because it can be drawn in the plane without crossing edges (see below).



B

3. Graph C:

(a) It has NO Euler trail because four of its vertices have even degree $(b, d, f, h.)$

(b) For the same reason it does NOT have an Euler circuit.

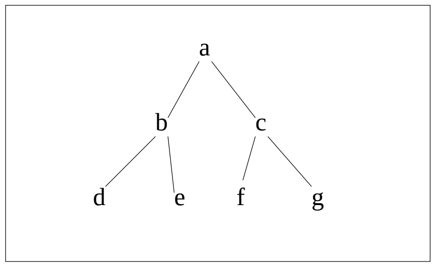
(c) It has a Hamilton path, e.g.: $a\,b\,c\,f\,e\,d\,g\,h\,i$.

(d) It has NO Hamilton cycle because the graph is bipartite, $V_1 = \{a, c, e, g, i\}$, $V_2 = \{b, d, f, h\}$, and $|V_1| = 5 \neq |V_2| = 4$.

(e) It is planar because it can be (and actually it already is) drawn in the plane without crossing edges.

10. (Trees) In the ordered rooted tree below the vertices are given the natural alphabetical order (a, b, c, d, e, f, g). Find the following orderings of the vertices:

1. Inorder transversal.
2. Preorder transversal.
3. Postorder transversal.



Solution:

1. Inorder transversal: d, b, e, a, f, c, g
2. Preorder transversal: a, b, d, e, c, f, g
3. Postorder transversal: d, e, b, f, g, c, a