CS 310 - Winter 2000 - Sample Final Exam 3

Last Name:	
First Name:	

1. (Logic) Prove that the following logical implications cannot be reversed:

1. $\exists x\, [p(x) \wedge q(x)] \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$.
2. $\forall x\, [p(x) \vee q(x)] \Leftarrow \forall x\, p(x) \vee \forall x\, q(x)$.

2. (Sets) Let X be a set with n elements, and assume that $a \in X$. How many elements are there in each of the following sets:

1. $\mathcal{P}(X) =$ power set of X?
2. $\mathcal{P}(X - \{a\})$?
3. $\mathcal{P}(X) - \{\{a\}\}$?
4. $\mathcal{P}(\{X, a\})$?

3. (Functions) Let $f : \{0, 1, 2, 3\} \to \{1, 2, 3, 4\}$ be the function $f(x) =$ "remainder of dividing 2^x by 5". Prove that f is invertible and find its inverse.

4. (Operations) Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operation on $\mathbb{R}^+$:

$$x \circ y = \frac{1}{\frac{1}{x} + \frac{1}{y}} .$$

5. (Relations) On $\mathbb{Z}^2$ we define the relation $(x_1, y_1)\, \mathcal{R}\, (x_2, y_2) \Leftrightarrow x_1 - y_1 = x_2 - y_2$. Prove that $\mathcal{R}$ is an equivalence relation. Describe the equivalence classes.

6. (Counting) In how many ways can we get an odd number of tails if we toss 7 (distinct) coins?

7. (Recursiveness) Find a close-form formula for the function $f : \mathbb{N} \to \mathbb{N}$ defined recursively in the following way:

1. $f(0) = 0$.
2. $f(n) = f(n-1) + n$, for $n \geq 1$.

8. (Modular Arithmetic) How many units are there in $(\mathbb{Z}_{45}, +, \cdot)$? How many zero divisors?

For each of the following elements $x \in \mathbb{Z}_{45}$, determine if x is a unit or a zero divisor. If it is a unit, find its inverse. If it is a zero divisor find another element y such that $x \cdot y = 0$ in $\mathbb{Z}_{45}$: (1) $x = 7$, (2) $x = 12$, (3) $x = 15$, (4) $x = 22$.

9. (Graphs, Counting, Trees) How many Hamiltonian paths beginning at vertex A are there in the graph shown in figure 1? How many Hamiltonian cycles? (Hint: Use a decision tree).

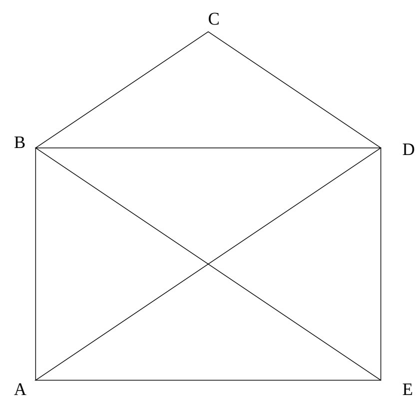

Figure 1: Graph G.

10. (Graphs, Coloring) Is the graph G shown in figure 1 planar? If so, draw it without intersecting edges. Find its chromatic number $\chi(G)$. Is it possible to remove an edge e from G so that $\chi(G - \{e\}) < \chi(G)$? Justify your answers.