CS 310 - Winter 2000 - Sample Final Exam 2

Last Name:	
First Name:	

1. (Logic) For each of the following statements find a model and a countermodel:

1. $\exists x \exists y \exists z\, [(x \neq y) \wedge (y \neq z) \wedge (x \neq z)]$.
2. $\exists x \exists y \forall z\, [(x = z) \vee (y = z)]$.
3. $\forall x \exists y\, (x = y^2)$.
4. $\forall x\, (y = z + x^2)$.

2. (Sets) Let A, B, C be the following sets: $A = \{(x, y) \in \mathbb{Z}^2 \mid 4x = 3y\}$, $B = \{(x, y) \in \mathbb{Z}^2 \mid y = 0\}$, $C = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = 25\}$. Find $A \cap B$, $A \cap C$, $B \cap C$, $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

3. (Functions) Find the largest subset $D \subseteq \mathbb{R}$ that can be the domain of a function $f : D \to R$ defined in the following way:

$$f(x) = \frac{1}{2 - \sqrt{x}}\,.$$

4. (Operations) We define the set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. What kind of structure (group, ring, field) is $(\mathbb{Q}[\sqrt{2}], +, \cdot)$?

5. (Relations) For each of the following subsets of $\mathbb{R}$ determine (if they exist) its least element, greatest element, glb and lub: $A = [3, 7)$, $B = \{1/n \mid n = 1, 2, 3 \ldots\}$, $C = \{2n \mid n \in \mathbb{Z}\}$, $D = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$.

6. (Counting) In how many ways can we go from point $(0, 0)$ to point $(4, 5)$ (given in Cartesian coordinates) if the only moves allowed are: $R =$ "go right 1 unit to the right", and $U =$ "go up by 1 unit"?

7. (Induction) Prove the following by induction:

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n} \qquad \text{for every } n \geq 1\,.$$

8. (Congruences) Solve the following system of congruences:

$$\begin{cases} x \equiv 2 \pmod{5} \\ x \equiv -2 \pmod{7} \end{cases}$$

9. (Graphs) In each of the following cases draw a connected loop-free planar graph (not a multigraph) with the given characteristics, or prove that none exists:

1. 4 vertices all of degree 3, 4 regions.
2. 4 vertices, 6 edges, 5 regions.
3. 4 vertices all of degree 4.
4. 6 vertices all of degree 3, 5 regions.

10. (Trees) How many edges does $K_{3,5}$ have? Draw a spanning tree for $K_{3,5}$. How many edges does the spanning tree have?