CS 310 - Winter 2000 - Sample Final Exam

Last Name:	
First Name:	

1. (Logic) Determine the truth value of each the following statements:

1. $[\exists x \forall y\, (x \leq y)] \vee [\exists x \forall y\, (y \leq x)]$
2. $[\exists x \forall y\, (x \leq y)] \wedge [\exists x \forall y\, (y \leq x)]$
3. $\exists x \forall y\, [(x \leq y) \vee (y \leq x)]$
4. $\exists x \forall y\, [(x \leq y) \wedge (y \leq x)]$

in each of the following universes of discourse: $\mathcal{U} = \{0, 1, 2, 3, 4, 5\}$, $\mathcal{U} = \mathbb{N}$, $\mathcal{U} = \mathbb{Z}$, $\mathcal{U} = \{x \in \mathbb{Z} \mid x \leq 0\}$.

2. (Sets) Let A, B, C be the following sets: $A = \{x \in \mathbb{Z} \mid \exists y \in \mathbb{Z}, y = x^2\}$, $B = \{x \in \mathbb{R} \mid x < 20\}$, $C = \{x \in \mathbb{R} \mid x > -5\}$. Find $\overline{A \cap B \cap C}$.

3. (Functions) Let $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Let $S : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$, $T : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ be the following functions: $S(x) = -1/x$, $T(x) = x + 1$ (assume $\infty + 1 = \infty$, $-1/\infty = 0$, $-1/0 = \infty$.) Find the following functions: $S^2(x)$, $S^{-1}(x)$, $T^2(x)$, $T^{-1}(x)$, $(S \circ T)(x)$, $(S \circ T)^2(x)$.

4. (Operations) Given a group $(G, *)$, a subgroup of G is any non-empty subset $H \subseteq G$ such that $(H, *)$ is a group. Prove that a non-empty subset $H \subseteq G$ is a subgroup of G if and only if for every $x, y \in H$, $x * y^{-1} \in H$.

5. (Relations) If H is a subgroup of $(G, *)$, prove that the relation $x \,\mathcal{R}\, y \Leftrightarrow x * y^{-1} \in H$ for every $x, y \in G$ is an equivalence relation.

6. (Counting) We have 3 Mathematics books, 4 Physics books and 5 Computer Science books. We want to put them on a shelf in such a way that the books of the same subject remain together. In how many ways can the books be put on the shelf with that restriction?

7. (Recurrences) Solve the following recurrence:

$$x_n = 5\, x_{n-1} - 6\, x_{n-2}\,; \qquad x_0 = 0,\ x_1 = 1\,.$$

8. (Divisibility) Solve the following Diophantine equation:

$$11\, x + 5\, y = 1\,.$$

9. (Graphs) The vertices and edges of a polyhedron define a graph. Which Platonic solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron) contain an Euler circuit? Why?

10. (Trees) Represent the following algebraic expression with a tree:

$$a * b + c * d \uparrow e\,.$$

Express it in Polish notation and in reversed Polish notation.