CS 310 - Winter 2000 - Midterm Exam (solutions)

SOLUTIONS

1. (Logic)

(a) Prove the following logical equivalence by using truth tables:

$$p \veebar (q \veebar r) \Leftrightarrow (p \veebar q) \veebar r\,,$$

where "$\veebar$" represents exclusive or.

(b) Find the negation of the following quantified statement, leaving the answer in prenex normal form and the statement inside in disjunctive normal form:

$$\forall x\,((x > 0) \wedge \exists y\,(y < x))\,.$$

Solution:

(a) The truth table is the following:

p	q	r	$p \veebar q$	$(p \veebar q) \veebar r$	$q \veebar r$	$p \veebar (q \veebar r)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	0	1	0
1	1	1	0	1	0	1

Since $p \veebar (q \veebar r)$ and $(p \veebar q) \veebar r$ have the same truth table, they are logically equivalent.

(b)

$$\neg\forall x\,((x > 0) \wedge \exists y\,(y < x)) \Leftrightarrow \neg\forall x \exists y\;((x > 0) \wedge (y < x)) \Leftrightarrow$$
$$\exists x \forall y\;\neg((x > 0) \wedge (y < x)) \Leftrightarrow \exists x \forall y\;(\neg(x > 0) \vee \neg(y < x)) \Leftrightarrow$$
$$\exists x \forall y\;((x \leq 0) \vee (y \geq x))\,.$$

2. (Sets) Assume that the universe of discourse is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let A, B, C be the following sets: $A = \{2x \mid x \in U\}$, $B = \{3x \mid x \in U\}$, $C = \{x \in U \mid \exists y \in U,\ y = x + 2\}$.

1. List the elements of A, B and C.

2. Find:

   $A \cup B =$

   $A \cap C =$

   $B \cap C =$

   $(A \cap C) \triangle (B \cap C) =$

   $A \triangle B =$

   $(A \triangle B) \cap C =$

Solution:

1. Since U is the universe of discourse, all the sets involved must be subsets of U:

   $A =$ (elements of U of the form $2x$ for $x \in U$) $= \{2, 4, 6, 8\}$

   $B =$ (elements of U of the form $3x$ for $x \in U$) $= \{3, 6, 9\}$

   $C = \{1, 2, 3, 4, 5, 6, 7\}$

   (I will also accept the answers $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$, $B = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$, and the corresponding ones for the second part of the problem.)

2. $A \cup B = \{2, 3, 4, 6, 8, 9\}$

   $A \cap C = \{2, 4, 6\}$

   $B \cap C = \{3, 6\}$

   $(A \cap C) \triangle (B \cap C) = \{2, 3, 4\}$

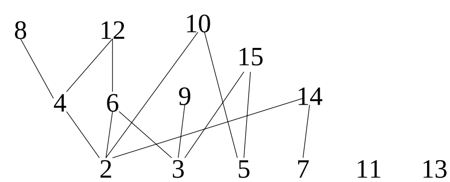
   $A \triangle B = \{2, 3, 4, 8, 9\}$

   $(A \triangle B) \cap C = \{2, 3, 4\}$

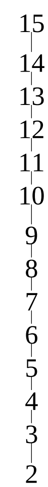
3. (Relations) Let A be the set $A = \{x \in \mathbb{Z} \mid 2 \leq x \leq 15\}$. Draw the Hasse diagram and find the largest, least, maximal and minimal elements of the following posets, if they exist:

1. $(A, |)$, where "$|$" represents the relation of divisibility.
2. $(A, \leq)$, where "$\leq$" represents the usual number inequality.

Solution:



(A,|)



(A,≤)

	largest	least	maximal	minimal
$(A, \|)$	does not exist	does not exist	$8, 9, 10, 11, 12, 13, 14, 15$	$2, 3, 5, 7, 11, 13$
$(A, \leq)$	15	2	15	2

4. (Functions) Let $f, g : \mathbb{R} \to \mathbb{R}$ be the following functions: $f(x) = 2x$, $g(x) = x+1$. Find $g \circ f$, $f \circ g$, f^{-1}, g^{-1}, $g^{-1} \circ f^{-1}$, $f^{-1} \circ g^{-1}$, $(g \circ f)^{-1}$ and $(f \circ g)^{-1}$.

Solution:

$$(g \circ f)(x) = g(f(x)) = f(x) + 1 = 2x + 1.$$

$$(f \circ g)(x) = f(g(x)) = 2g(x) = 2(x + 1) = 2x + 2.$$

$$(f^{-1})(x) = x/2.$$

$$(g^{-1})(x) = x - 1.$$

$$(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x)) = f^{-1}(x) - 1 = \frac{x}{2} - 1.$$

$$(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = \frac{g^{-1}(x)}{2} = \frac{x - 1}{2}.$$

$$(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x) = \frac{x - 1}{2}.$$

$$(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x) = \frac{x}{2} - 1.$$

5. (Operations) Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operation on $\mathbb{Z}$:

$$a * b = a + b + ab\,.$$

Justify your answer.

Solution:

1. It is commutative:

$$\begin{aligned} a * b &= a + b + ab\,, \\ b * a &= b + a + ba\,, \end{aligned}$$

   hence $a * b = b * a$.

2. It is associative:

$$\begin{aligned} a * (b * c) &= a * (b + c + bc) = a + (b + c + bc) + a\,(b + c + bc) \\ &= a + b + c + bc + ab + ac + abc\,, \\ (a * b) * c &= (a + b + ab) * c = (a + b + ab) + c + (a + b + ab)\,c \\ &= a + b + ab + c + ac + bc + abc\,, \end{aligned}$$

   hence $a * (b * c) = (a * b) * c$.

3. The identity element is 0:

$$a * 0 = 0 * a = a + 0 + 0 \cdot 0 = a\,.$$

4. There is no inverse element in general:

$$a * a' = a + a' + aa' = 0 \quad \Rightarrow \quad a' = -\frac{a}{1 + a}\,,$$

   hence the only invertible elements are 0 and -2.

6. (Counting) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) How many subsets of A contain exactly 2 even numbers and 3 odd numbers?

(b) How many permutations of size 5 of the elements of A contain exactly 2 even numbers and 3 odd numbers?

Solution:

(a) Set A has 4 even numbers, and we can select 2 even numbers from it in $\binom{4}{2} = 6$ ways. On the other hand there are 5 odd numbers in A, and we can select 3 of them in $\binom{5}{3} = 10$ ways. By the product rule we can select 2 even numbers and 3 odd numbers in $6 \times 10 = 60$ ways.

(b) We can generate the desired permutations by selecting 2 even numbers and 3 odd numbers, which can be done in 60 ways, then permuting those 5 elements in $5! = 120$ ways. By the product rule, the answer is $60 \cdot 120 = 7200$.