# CS 310 - Winter 2000 - Final Exam (solutions)

## SOLUTIONS

**1.** (Logic) Determine the truth value of each of the following statements:

S1: $\exists x \exists y \exists z \, (x < y \wedge y < z)$

S2: $\forall x \forall y \, [x < y \rightarrow \exists z \, (x < z \wedge z < y)]$

S3: $\exists x \forall y \, (x \neq y \rightarrow x < y)$

S4: $\exists x \forall y \, (x \neq y \rightarrow y < x)$

S5: $[\exists x \forall y \, (x \leq y)] \veebar [\exists x \forall y \, (y \leq x)]$

in the universe of discourse indicated by the header of each column of the following table (write your answers in the table):

*Solution:*

| | $\{0,1\}$ | $\mathbb{N}$ | $\mathbb{Z}$ | $\mathbb{Q}$ |
|---|---|---|---|---|
| S1 | 0 | 1 | 1 | 1 |
| S2 | 0 | 0 | 0 | 1 |
| S3 | 1 | 1 | 0 | 0 |
| S4 | 1 | 0 | 0 | 0 |
| S5 | 0 | 1 | 0 | 0 |

**2.** (Sets) Let $A, B, C$ be the following sets:

$$\begin{array}{l} A = \{(x,y) \in \mathbb{Q}^2 \mid y = x^2\} \\ B = \{(x,y) \in \mathbb{Q}^2 \mid y = x+2\} \\ C = \{(x,y) \in \mathbb{Q}^2 \mid y = 2\} \\ D = \{(x,y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 20\} \end{array}$$

Find each of the following sets:

1. $A \cap B$
2. $A \cap C$
3. $B \cap C$
4. $A \cap D$
5. $C \cap D$
6. $A \cap (B \cup D)$
7. $A \cap D \cap (B \cup C)$

*Solution:*

1. $A \cap B = \{(-1,1),(2,4)\}$
2. $A \cap C = \emptyset$
3. $B \cap C = \{(0,2)\}$
4. $A \cap D = \{(2,4),(-2,4)\}$
5. $C \cap D = \{(4,2),(-4,2)\}$
6. $A \cap (B \cup D) = (A \cap B) \cup (A \cap D) = \{(-1,1),(2,4),(-2,4)\}$
7. $A \cap D \cap (B \cup C) = (A \cap D \cap B) \cup (A \cap D \cap C) = \{(2,4)\} \cup \emptyset = \{(2,4)\}$

**3.** (Relations) Let $\mathcal{R}$ be the following relation on $\mathbb{R}$:

$$x \,\mathcal{R}\, y \Leftrightarrow x - y \in \mathbb{Z}\,.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.
2. Find the set $[50/3] \cap [0,1)$.[1]

*Solution:*

1. The relation is:
   (a) Reflexive: $x - x = 0 \in \mathbb{Z} \Rightarrow x \,\mathcal{R}\, x$.
   (b) Symmetric: $x \,\mathcal{R}\, y \Rightarrow x - y = n \in \mathbb{Z} \Rightarrow y - x = -n \in \mathbb{Z} \Rightarrow y \,\mathcal{R}\, x$.
   (c) Transitive: If $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, z$ then $x - y = n \in \mathbb{Z}$ and $y - z = m \in \mathbb{Z}$, so $x - z = n + m \in \mathbb{Z}$, hence $x \,\mathcal{R}\, y$.

   Hence $\mathcal{R}$ is an equivalence relation.
2. $[50/3] \cap [0,1) = \{2/3\}$

[1] $[x]$ = equivalence class of $x$; $[0,1) = \{r \in \mathbb{R} \mid 0 \leq r < 1\}$.

**4.** (Functions) Consider the functions $f, g, h : \{0,1,2\} \to \{0,1,2\}$ defined as $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, $g(0) = 1$, $g(1) = 0$, $g(2) = 2$, $h(0) = 0$, $h(1) = 2$, $h(2) = 1$.

1. Show that $h \circ f = f \circ g$.
2. Show that $f^3 = \text{id}$ on $\{0,1,2\}$.
3. Write $h$ as a suitable composition of $f$ and $g$.

*Solution:*

1. $h(f(0)) = h(1) = 2,\ h(f(1)) = h(2) = 1,\ h(f(2)) = h(0) = 0.$
   $f(g(0)) = f(1) = 2,\ f(g(1)) = f(0) = 1,\ f(g(2)) = f(2) = 0.$
2. $f^3(0) = f(f(f(0))) = f(f(1)) = f(2) = 0 = \text{id}(0).$
   $f^3(1) = f(f(f(1))) = f(f(2)) = f(0) = 1 = \text{id}(1).$
   $f^3(2) = f(f(f(2))) = f(f(0)) = f(1) = 2 = \text{id}(2).$
3. There are many right answers, for instance: $h = g \circ f$, $h = f \circ g \circ f^2$, $h = f \circ g \circ f^{-1}$, $h = f^2 \circ g$.

**5.** (Operations) Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operation on $\mathbb{R}^+ \cup \{0\}$:

$$x \circ y = \sqrt{x^2 + y^2}\,.$$

Justify your answer.

*Solution:*

1. It is commutative:

$$x \circ y = \sqrt{x^2 + y^2} = \sqrt{y^2 + x^2} = y \circ x\,.$$

2. It is associative:

$$(x \circ y) \circ z = \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} = \sqrt{x^2 + y^2 + z^2}\,,$$
$$x \circ (y \circ z) = \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} = \sqrt{x^2 + y^2 + z^2}\,,$$

hence $(x \circ y) \circ z = x \circ (y \circ z)$.

3. The identity element is 0:

$$x \circ 0 = 0 \circ x = \sqrt{x^2 + 0^2} = \sqrt{x^2} = x\,.$$

4. There is no inverse:

$$x \circ x' = 0 \Rightarrow \sqrt{x^2 + x'^2} = 0 \Rightarrow x^2 + x'^2 = 0 \Rightarrow x = x' = 0\,,$$

so the only invertible element is 0.

**6.** (Counting) In how many ways can we get a total of 8 points by throwing a die three times? Example: one way is 2 points on the first throw, 3 points on the second throw, 3 points on the third throw (note that the order is relevant).

*Solution:*

The answer is the number of integer solutions to the following equation:

$$x_1 + x_2 + x_3 = 8$$

with the restrictions $1 \leq x_1, x_2, x_3 \leq 6$. Calling $x'_i = x_i + 1$ we get that the problem is equivalent to counting the number of integer solutions to the equation

$$x'_1 + x'_2 + x'_3 = 5$$

with the restrictions $0 \leq x'_1, x'_2, x'_3 \leq 5$. Since the sum must be 5, the restriction $x'_1, x'_2, x'_3 \leq 5$ is superfluous, so the solution is $P(3+5-1; 5, 3-1) = P(7; 5, 2) = 21$.

**7.** (Recurrences) Solve the following recurrence:

$$x_n = x_{n-1} + 2\,x_{n-2}\,; \qquad x_0 = 0,\ x_1 = 3\,.$$

*Solution:*

The characteristic equation is

$$r^2 - r - 2 = 0\,.$$

The characteristic roots are $r_1 = -1$ and $r_2 = 2$. The general solution to the recurrence is

$$x_n = A\,(-1)^n + B\,2^n\,.$$

Using the initial condition we get

$$\begin{cases} A + B = 0 \\ -A + 2B = 3 \end{cases}$$

From here we get $A = -1$, $B = 1$, hence:

$$x_n = -(-1)^n + 2^n\,.$$

**8.** (Divisibility) Solve the following Diophantine equation:

$$23\,x + 10\,y = 1\,.$$

*Solution:*

Using the Euclidean algorithm:

$$\begin{aligned} 23 &= 2 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \end{aligned}$$

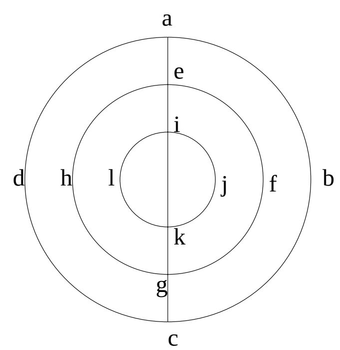
Hence:

$$1 = 10 - 3 \cdot 3 = 10 - 3 \cdot (23 - 2 \cdot 10) = -3 \cdot 23 + 7 \cdot 10\,.$$

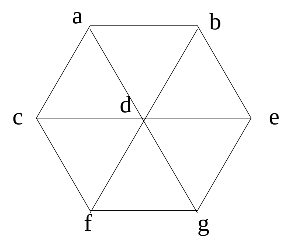
So, $(x_0, y_0) = (-3, 7)$ is a particular solution. The general solution is:

$$\begin{aligned} x &= -3 + 10k \\ y &= \phantom{-}7 - 23k \end{aligned}$$

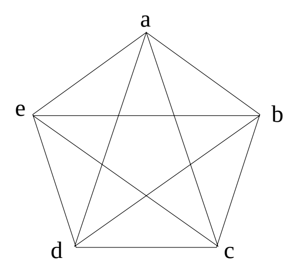
**9.** (Graphs) For each of the following graphs, find an Euler circuit, or an Euler trail, or prove that there is no Euler circuit or trail.




(A)




(B)




(C)

*Solution:*

(A) There is an Euler trail, for instance: $a\,b\,c\,d\,a\,e\,f\,g\,h\,e\,i\,j\,k\,l\,i\,k\,g\,c$.

(B) There is no Euler circuit or trail, because the graph has 6 points with odd degree.

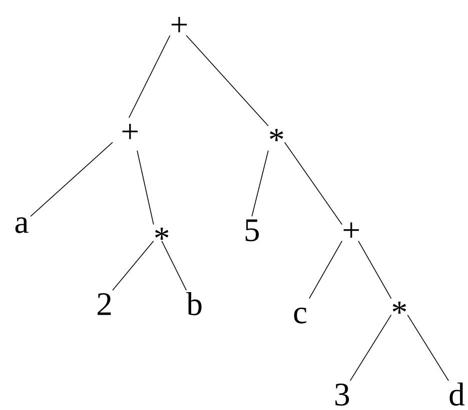
(C) There is an Euler circuit, for instance: $a\,b\,c\,d\,e\,a\,c\,e\,b\,d\,a$.

**10.** (Trees) Represent the following algebraic expression with a tree:

$$a + 2 * b + 5 * (c + 3 * d)$$

Express it in Polish notation and in reversed Polish notation.

*Solution:*




Polish notation:

$$+ \; + a \; * \; 2 \, b \; * \; 5 \; + \; c \; * \; 3 \, d$$

Reversed Polish notation:

$$a \, 2 \, b \; * \; + 5 \, c \, 3 \, d \; * \; + \; * \; +$$