CS 310 - Spring 2000 - Midterm Exam (solutions)

SOLUTIONS

1. (Logic)

1. Prove the following logical equivalence by using Laws of Logic (Algebra of Propositions):

$$(p \wedge q) \rightarrow r \;\Leftrightarrow\; (p \rightarrow r) \vee (q \rightarrow r)\,.$$

   (Assume that '$\rightarrow$' is defined by "$p \rightarrow q \Leftrightarrow \neg p \vee q$".)

2. For each of the following quantified statements find a model and a countermodel (if any exists):

   (a) $\exists x \exists y \forall z\,[(z = x) \vee (z = y)]$.

   (b) $\exists x \exists y \exists z\,[(x \neq y) \wedge (x \neq z) \wedge (y \neq z)]$.

   (c) $\forall x \exists y\,(x = y)$.

   (d) $\exists x \forall y\,(x = y)$.

Solution:

1. $(p \wedge q) \rightarrow r \overset{\text{(Def. of '}\rightarrow\text{')}}{\Longleftrightarrow} \neg(p \wedge q) \vee r \overset{\text{(DeMorgan's)}}{\Longleftrightarrow} \neg p \vee \neg q \vee r \overset{\text{(Idempotent)}}{\Longleftrightarrow} \neg p \vee \neg q \vee r \vee r \overset{\text{(Associative)}}{\Longleftrightarrow} (\neg p \vee r) \vee (\neg q \vee r) \overset{\text{(Def. of '}\rightarrow\text{')}}{\Longleftrightarrow} (p \rightarrow r) \vee (q \rightarrow r)$.

2. (a) Model: $\{0, 1\}$ (or any set with at most two elements).
   Countermodel: $\mathbb{Z}$ (or any set with more than two elements).

   (b) Model: $\{0, 1, 2\}$ (or any set with at least three elements).
   Countermodel: $\{0\}$ (or any set with less than three elements).

   (c) Model: Any (the statement is always true).
   Countermodel: There is none.

   (d) Model: $\{0\}$ (or any set with exactly one element).
   Countermodel: $\mathbb{Z}$ (or any set with at least two elements).

2. (Sets) Let A, B, C, be the following sets: $A = \{a, b, c\}$, $B = \{x \in \mathbb{Z} \mid 0 \leq x < 3\}$, $C = \{x \in Z \mid 0 < x \leq 3\}$. Find the following sets (list their elements):

1. $B \cap C =$
2. $A \times (B \cap C) =$
3. $A \times B =$
4. $A \times C =$
5. $(A \times B) \cap (A \times C) =$

Solution:

1. $B \cap C = \{1, 2\}$
2. $A \times (B \cap C) = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$
3. $A \times B = \{(a, 0), (b, 0), (c, 0), (a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$
4. $A \times C = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2), (a, 3), (b, 3), (c, 3)\}$
5. $(A \times B) \cap (A \times C) = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

3. (Operations) On $\mathbb{R}^2$ we define the following operation:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\,.$$

1. Prove that $(\mathbb{R}^2, +)$ is a commutative group.
2. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined $f(x, y) = x$ is a group-homomorphism from $(\mathbb{R}^2, +)$ to $(\mathbb{R}, +)$.

Solution:

1. (a) Commutative property:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = \\ (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1)\,.$$

(b) Associative property:

$$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = \\ (x_1 + x_2, y_1 + y_2) + (x_3, y_3) = (x_1 + x_2 + x_3, y_1 + y_2 + y_3)\,,$$

$$(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] = \\ (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1 + x_2 + x_3, y_1 + y_2 + y_3)\,.$$

Hence:

$$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]\,.$$

(c) Identity element. The identity element is $(0, 0)$:

$$(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)\,.$$

(d) Inverse element: The inverse element of (x, y) is $(-x, -y)$:

$$(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)\,.$$

2. $f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = x_1 + x_2$.
$f(x_1, y_1) + f(x_2, y_2) = x_1 + x_2$.
Hence: $f((x_1, y_1) + (x_2, y_2)) = f(x_1, y_1) + f(x_2, y_2)$.

4. (Relations) On $\mathbb{R}^2$ we define the relation

$$(x_1, y_1)\,\mathcal{R}\,(x_2, y_2) \Leftrightarrow 2x_1 + 3y_1 = 2x_2 + 3y_2\,.$$

Prove that $\mathcal{R}$ is an equivalence relation.

Solution:

1. Reflexive: $2x + 3y = 2x + 3y \Rightarrow (x, y)\,\mathcal{R}\,(x, y)$.
2. Symmetric: $(x_1, y_1)\,\mathcal{R}\,(x_2, y_2) \Rightarrow 2x_1 + 3y_1 = 2x_2 + 3y_2 \Rightarrow 2x_2 + 3y_2 = 2x_1 + 3y_1 \Rightarrow (x_2, y_2)\,\mathcal{R}\,(x_1, y_1)$
3. Transitive:

$$\left.\begin{array}{l}(x_1, y_1)\,\mathcal{R}\,(x_2, y_2) \Rightarrow 2x_1 + 3y_1 = 2x_2 + 3y_2 \\ (x_2, y_2)\,\mathcal{R}\,(x_3, y_3) \Rightarrow 2x_2 + 3y_2 = 2x_3 + 3y_3\end{array}\right\} \Rightarrow$$

$$2x_1 + 3y_1 = 2x_3 + 3y_3 \Rightarrow (x_1, y_1)\,\mathcal{R}\,(x_3, y_3)\,.$$

5. (Functions) Let A be the set $A = \{0, 1, 2, 3, 4\}$. Let $f, g : A \to A$ be defined in the following way: $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 0$; $g(0) = 1$, $g(1) = 0$, $g(2) = 2$, $g(3) = 3$, $g(4) = 4$. A convenient way of representing those functions consists of listing between parenthesis the images of $0, 1, 2, 3, 4$ in this order, so that $f = (1, 2, 3, 4, 0)$, $g = (1, 0, 2, 3, 4)$.

1. Find $g \circ f$, $f \circ g$, f^{-1}, g^{-1}, $(g \circ f)^{-1}$, $(f \circ g)^{-1}$.
2. Let $h : A \to A$ be the function $h = (0, 2, 1, 3, 4)$. Find $h \circ f$ and $f \circ h$. Compare to the previously found compositions and write h as a suitable composition of f and g.

Solution:

1. $g \circ f = (0, 2, 3, 4, 1)$.
   $f \circ g = (2, 1, 3, 4, 0)$.
   $f^{-1} = (4, 0, 1, 2, 3)$.
   $g^{-1} = (1, 0, 2, 3, 4)$.
   $(g \circ f)^{-1} = (0, 4, 1, 2, 3)$.
   $(f \circ g)^{-1} = (4, 1, 0, 2, 3)$.
2. $h \circ f = (2, 1, 3, 4, 0)$.
   $f \circ h = (1, 3, 2, 4, 0)$.
   $h \circ f = f \circ g$, hence: $h = f \circ g \circ f^{-1}$.

6. (Counting)

(a) How many non negative integer solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 = 10\,.$$

(b) How many of those solutions are strictly positive?

(c) How many non negative solutions consists of even numbers only?

Solution:

(a) $\binom{4+10-1}{10} = \binom{13}{10} = 286$.

(b) Calling $x_1 = y_1 + 1$, $x_2 = y_2 + 1$, $x_3 = y_3 + 1$, $x_4 = y_4 + 1$, the equation becomes:

$$y_1 + y_2 + y_3 + y_4 = 10 - 4 = 6\,.$$

Its non negative solutions correspond to strictly positive solutions to the original equation. Their number is $\binom{4+6-1}{6} = \binom{9}{6} = 84$.

(c) Calling $x_1 = 2z_1$, $x_2 = 2z_2$, $x_3 = 2z_3$, $x_4 = 2z_4$, the equation becomes:

$$z_1 + z_2 + z_3 + z_4 = 5\,.$$

Its non negative solutions correspond to even non negative solutions to the original equation. Their number is $\binom{4+5-1}{5} = \binom{8}{5} = 56$.