CS 310 - Spring 2000 - Final Exam (solutions)

SOLUTIONS

1. (Logic) Determine the truth value of each of the following statements:

S1: $\exists x\,(x = 0)$

S2: $\forall x \exists y\,(x + 1 = y)$

S3: $\neg \exists x\,(x + 1 = 0)$

S4: $\forall x \forall y\,[\{\exists z\,(z = x + 1 \wedge z = y + 1)\} \rightarrow (x = y)]$

S5: $\forall x\,\{(x \leq 0) \rightarrow [\forall y\,(x \leq y)]\}$

S6: $\forall x\,(x^2 = x)$

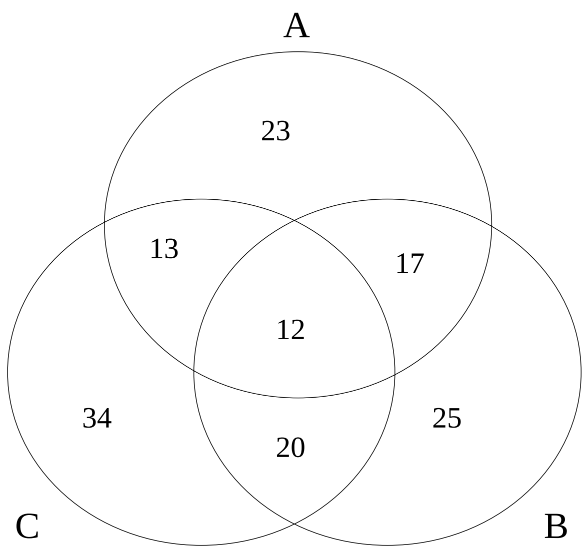
in the universe of discourse indicated by the header of each column of the following table (write your answers in the table):

Solution:

	$\{0,1\}$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$
S1	1	1	1	1
S2	0	1	1	1
S3	1	1	0	0
S4	1	1	1	1
S5	1	1	0	0
S6	1	0	0	0

2. (Venn Diagrams) Let A, B, C be the three sets such that $|A| = 65$, $|B| = 74$, $|C| = 79$, $|A \cap B| = 29$, $|A \cap C| = 25$, $|B \cap C| = 32$, $|A \cup B \cup C| = 144$. Draw their Venn diagram and count the number of elements in each of the seven small regions determined by it, i.e.: $A \cap B \cap C$, $A \cap B \cap \overline{C}$, $A \cap \overline{B} \cap C$, $\overline{A} \cap B \cap C$, $A \cap \overline{B} \cap \overline{C}$, $\overline{A} \cap B \cap \overline{C}$, $\overline{A} \cap \overline{B} \cap C$.

Solution:



3. (Relations) Let $\mathcal{R}$ be the following relation on $\mathbb{R}$:

$$x \,\mathcal{R}\, y \Leftrightarrow y - x \in \mathbb{Q}^+ \cup \{0\}\,.$$

1. Prove that $\mathcal{R}$ is a partial order.

2. Prove that the order is not total.

Solution:

1. The relation is:

   (a) Reflexive: $x - x = 0 \in \mathbb{Q}^+ \cup \{0\} \Rightarrow x \,\mathcal{R}\, x$.

   (b) Antisymmetric:
   We have:

   $$x \,\mathcal{R}\, y \Rightarrow y - x \in \mathbb{Q}^+ \cup \{0\} \Rightarrow x \leq y\,,$$

   and

   $$y \,\mathcal{R}\, x \Rightarrow x - y \in \mathbb{Q}^+ \cup \{0\} \Rightarrow y \leq x\,,$$

   hence if $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, x$ then $x \leq y$ and $y \leq x$, which implies $x = y$.

   (c) Transitive: If $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, z$ then $x - y = r \in \mathbb{Q}^+ \cup \{0\}$ and $y - z = s \in \mathbb{Q}^+ \cup \{0\}$, so $x - z = r + s \in \mathbb{Q}^+ \cup \{0\}$, hence $x \,\mathcal{R}\, z$.

   Hence $\mathcal{R}$ is an equivalence relation.

2. For instance, 0 and $\sqrt{2}$ are non comparable because $\sqrt{2} - 0 = \sqrt{2} \notin \mathbb{Q}^+ \cup \{0\}$ and $0 - \sqrt{2} = -\sqrt{2} \notin \mathbb{Q}^+ \cup \{0\}$, hence $\sqrt{2} \not\mathcal{R}\, 0$ and $0 \not\mathcal{R} \sqrt{2}$.

4. (Functions) Consider the functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (2x + y, x + y) ,$$
$$g(x, y) = (3x + y, x - y) .$$

Find:

1. $g \circ f$.
2. $f \circ g$.
3. f^{-1}. [Hint: write $f^{-1}(x, y) = (ax+by, cx+dy)$ and use $(f^{-1}\circ f)(x, y) = (x, y)$ in order to determine a, b, c, d.]

Solution:

1. $(g \circ f)(x, y) = g(f(x, y)) = g(2x + y, x + y) = (7x + 4y, x)$.
2. $(f \circ g)(x, y) = f(g(x, y)) = f(3x + y, x - y) = (7x + y, 4x)$.
3. Let $f^{-1}(x, y) = (ax + by, cx + dy)$. Then

   $$(f^{-1} \circ f)(x, y) = ((2a + b)\, x + (a + b)\, y, (2c + d)\, x + (c + d)\, y) = (x, y) .$$

   Hence

   $$\begin{cases} (2a + b)\, x + (a + b)\, y = x \\ (2c + d)\, x + (c + d)\, y = y \end{cases}$$

   so:

   $$\begin{cases} 2a + b = 1 \\ a + b = 0 \\ 2c + d = 0 \\ c + d = 1 \end{cases}$$

   which implies $a = 1,\ b = -1,\ c = -1,\ d = 2$.

   Thus, f^{-1} must be: $f^{-1}(x, y) = (x - y, -x + 2y)$.

5. (Operations) Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operation on $\mathbb{R}^+ \cup \{0\}$:

$$x \circ y = |x - y| \,,$$

where $|x|$ = absolute value of x. Justify your answer.

Solution:

1. It is commutative:

$$x \circ y = |x - y| = |y - x| = y \circ x \,.$$

2. It is NOT associative, for instance:

$$1 \circ (2 \circ 3) = 1 \circ |2 - 3| = 1 \circ 1 = |1 - 1| = 0 \,,$$
$$(1 \circ 2) \circ 3 = |1 - 2| \circ 3 = 1 \circ 3 = |1 - 3| = 2 \,,$$

hence $1 \circ (2 \circ 3) \neq (1 \circ 2) \circ 3$.

3. The identity element is 0:

$$x \circ 0 = 0 \circ x = |x - 0| = |x| = x \,.$$

4. Every x is its own inverse:

$$x \circ x = |x - x| = 0 \,.$$

6. (Counting) Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be a set with 10 elements. In how many ways can we partition A into two disjoint non-empty subsets whose union is A? Note that the order of the subsets is irrelevant, for instance $A = \{0, 2, 4\} \cup \{1, 3, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 6, 7, 8, 9\} \cup \{0, 2, 4\}$ are considered one and the same partition.

Solution:

For each subset A' of A, if $\overline{A'} = A - A'$ then $A = A' \cup \overline{A'}$ and $A' \cap \overline{A'} = \emptyset$. The problem requires $A' \neq \emptyset$ and $\overline{A'} \neq \emptyset$, so the subset A' can be chosen in $|\mathcal{P}(A)| - 2 = 2^{10} - 2 = 1022$ ways. Since $A' \cap \overline{A'} = \overline{A'} \cap A'$ we get $1022/2 = 511$ partitions. So the answer is 511.

7. (Recurrences) Solve the following recurrence:

$$x_n = 6\ x_{n-1} - 9\,x_{n-2}\,; \qquad x_0 = 0,\ x_1 = 3\,.$$

Solution:

The characteristic equation of the recurrence is

$$r^2 - 6\,r + 9 = 0\,.$$

It has a double root $r = 3$, so its general solution is

$$x_n = A\,3^n + B\,n\,3^n\,.$$

Using the initial conditions we get

$$\begin{cases} A & & = 0 \\ 3A + 3B & & = 3 \end{cases}$$

From here we get $A = 0$, $B = 1$, hence:

$$x_n = n\,3^n\,.$$

8. (Divisibility) Find $g = \gcd(120, 210)$ and solve the Diophantine equation:

$$120\,x + 210\,y = g\,.$$

Solution:

Using the Euclidean algorithm:

$$\begin{array}{rcl} 210 & = & 1 \cdot 120 + 90 \\ 120 & = & 1 \cdot 90 + 30 \\ 90 & = & 3 \cdot 30 + 0 \end{array}$$

Hence $g = \gcd(210, 120) = 30$, and:

$$30 = 120 - 90 = 120 - (210 - 120) = 2 \cdot 120 - 210\,.$$

So, $(x_0, y_0) = (2, -1)$ is a particular solution.

The solution to the homogeneous equation

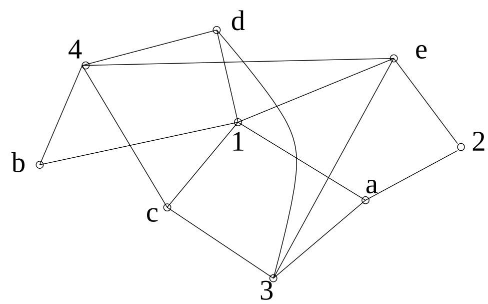
$$120\,x + 210\,y = 0$$

is $(x_h, y_h) = (210k/g, -120k/g) = (7k, -4k)$, hence the general solution is:

$$\left\{ \begin{array}{rcccr} x & = & x_0 + x_h & = & 2 + 7k \\ y & = & y_0 + y_h & = & -1 - 4k \end{array} \right.$$

where $k \in \mathbb{Z}$.

9. (Graphs) Consider the following graph G:



Answer four of the following six questions:

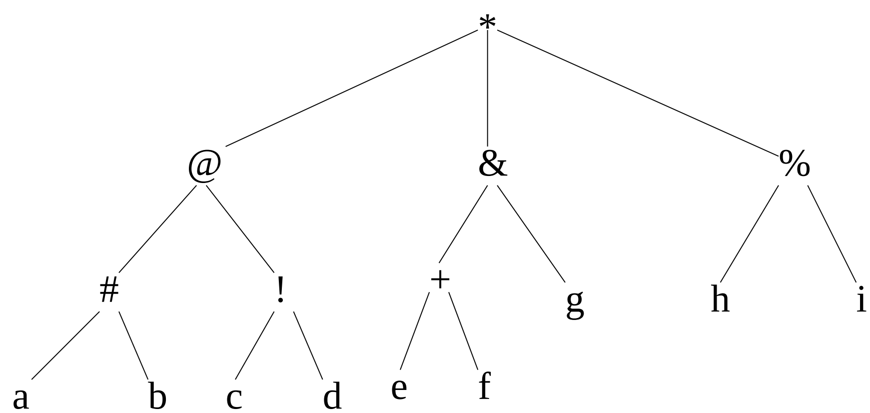
1. Prove that G is bipartite, i.e., find a partition of the set of vertices into two subsets V_1, V_2 such that all edges connect vertices in V_1 to vertices in V_2.
2. Does it have a Hamilton path?
3. Does it have a Hamilton cycle?
4. Does it have an Euler circuit?
5. Does it have an Euler trail (not a circuit)?
6. Is it planar?

Justify your answers.

Solution:

1. $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{a, b, c, d, e\}$.
2. Yes, there is a Hamilton path, e.g.: $b\,4\,d\,3\,c\,1\,a\,2\,e$.
3. No, there is no Hamilton cycle, because G it is bipartite (see question 1) but $|V_1| \neq |V_2|$ (note that $|V_1| + |V_2| = 9$, and odd number.)
4. No, because it has 4 vertices with odd degree (it should have none).
5. No, because it has 4 vertices with odd degree (it should have 2).
6. No, because the subgraph induced by $\{1, 3, 4, c, d, e\}$ is isomorphic to $K_{3,3}$.

10. (Trees) Consider the following rooted tree:



1. Find its *Preorder Transversal.*
2. Find its *Postorder Transversal.*
3. For the binary subtree with root "@" find its *Inorder Transversal.*

Solution:

1. Preorder Transversal:

$$* \; @ \; \# \; a \; b \; ! \; c \; d \; \& \; + \; e \; f \; g \; \% \; h \; i$$

2. Postorder Transversal:

$$a \; b \; \# \; c \; d \; ! \; @ \; e \; f \; + \; g \; \& \; h \; i \; \% \; *$$

3. Inorder Transversal of subtree:

$$a \; \# \; b \; @ \; c \; ! \; d$$