Math B17 - Winter 1999 - Midterm Exam No. 2 (solutions)

SOLUTIONS

1. Let A, B and C be the following matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

Compute $(3A + B)\,C$.

Solution:

$$\begin{aligned}(3A + B)\,C &= \left(\begin{bmatrix} 3 & 12 \\ 9 & 18 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} \right) \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 9 \\ 10 & 23 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\end{aligned}$$

2. Is the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the column space of the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 3 & 2 \end{bmatrix}$?

Solution:

The vector $\mathbf{b}$ is in the column space of A iff the system $A\mathbf{v} = \mathbf{b}$ has a solution. The augmented matrix is:

$$A' = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 2 \\ 1 & 3 & 2 & 1 \end{array}\right]$$

After using Gauss-Jordan reduction it becomes: $\left[\begin{array}{ccc|c} 1 & 3 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$

Here we see that $\operatorname{rank} A = \operatorname{rank} A' = 2$, hence the system has solution and the vector $\mathbf{b}$ does belong to the column space of the matrix A.

3. Solve the following system of equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 1 \end{cases}$$

Solution:

The augmented matrix is: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right]$

After using Gauss-Jordan reduction we get: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

i.e.:

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 1 \end{cases}$$

The solution is $x_1 = x_3$, $x_2 = 1 - 2x_3$, or:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

4. Find a basis and the dimension of the solution space for the system:

$$\begin{cases} x_1 \qquad - 3x_3 + x_4 - x_5 = 0 \\ \qquad x_2 - x_3 + 3x_4 + x_5 = 0 \end{cases}$$

Solution:

The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & -1 \\ 0 & 1 & -1 & 3 & 1 \end{bmatrix}$$

Note that A is already in Gauss-Jordan reduced form. Hence, the general solution of $A\mathbf{x} = \mathbf{0}$ is:

$$\begin{aligned} x_1 &= 3x_3 - x_4 + x_5 \\ x_2 &= x_3 - 3x_4 - x_5 \end{aligned}$$

and in matrix form: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the following set is a basis of the solution space:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and its dimension is 3.

5. Find the inverse of the following matrix: $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution:

$$A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

6. Find the eigenvalues and eigenvectors of the following matrix: $A = \begin{bmatrix} 3 & 4 & 4 \\ 2 & 1 & -2 \\ -4 & -4 & -1 \end{bmatrix}$.

Solution:

The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 4 & 4 \\ 2 & 1-\lambda & -2 \\ -4 & -4 & -1-\lambda \end{bmatrix} = -3 + 3\lambda^2 + \lambda - \lambda^3$$

$$= -(\lambda - 1)(\lambda - 3)(\lambda + 1)$$

Its roots are $\lambda = 3$, $\lambda = 1$ and $\lambda = -1$.

For $\lambda = 3$ we get $A - 3I = \begin{bmatrix} 0 & 4 & 4 \\ 2 & -2 & -2 \\ -4 & -4 & -4 \end{bmatrix}$

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The solutions of $(A - 3I)\,\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So we can take the following eigenvector: $\mathbf{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda = 1$ we get $A - I = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 0 & -2 \\ -4 & -4 & -2 \end{bmatrix}$

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

The solutions of $(A - I)\,\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} x_3 \\ -\frac{3}{2}\,x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

So we can take the following eigenvector: $\mathbf{v_2} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$

For $\lambda = -1$ we get $A + I = \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & -2 \\ -4 & -4 & 0 \end{bmatrix}$

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The solutions of $(A + I)\,\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

So we can take the following eigenvector: $\mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

As a summary, we get the following set of eigenvalues with their associated eigenvectors:

$$\lambda_1 = 3, \quad \mathbf{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 1, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}; \quad \lambda_3 = -1, \quad \mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$