Math B17 - Spring 1999 - Final Exam (solutions)

SOLUTIONS

1. Determine if the sequence $\{a_n\}$ converges, and find its limit if it does converge:

(a) $a_n = \dfrac{\sin n}{3^n}$

Solution:

We have: $-\dfrac{1}{3^n} \leq \dfrac{\sin n}{3^n} \leq \dfrac{1}{3^n}$. Hence, by the Squeeze Law it converges and

$$\lim_{n\to\infty} a_n = 0.$$

(b) $a_n = \dfrac{\ln 2n}{\ln 3n}$

Solution:

By L'Hôpital: $\displaystyle\lim_{n\to\infty} \frac{\ln 2n}{\ln 3n} = \lim_{n\to\infty} \frac{2/2n}{3/3n} = 1.$

(c) $a_n = \sqrt[n]{n}$.

Solution:

We have $\ln a_n = \ln n/n$. By L'Hôpital:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 1,$$

hence

$$\lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1.$$

2. Determine if the following infinite series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$

Solution:

It *converges*, because it is an alternating series and the n-th term is decreasing and tends to zero.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{2}$

Solution:

We have

$$\lim_{n\to\infty} \sqrt[n]{2} = 1$$

so the series *diverges* because the n-th term does not tend to zero.

3. Using the power series for $\ln(1 \pm x)$, find the power series for: $\ln \dfrac{1+x}{1-x}$.

Solution:

We have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Then:

$$\begin{aligned}
\ln \frac{1+x}{1-x} &= \ln(1+x) - \ln(1-x) \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \right) \\
&= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) \\
&= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

4. Solve the following system of equations or show that it has no solutions:

$$\begin{cases} x_1 + 2\,x_2 + 4\,x_3 = 3 \\ x_1 + 3\,x_2 + 9\,x_3 = 6 \\ 2\,x_1 + 5\,x_2 + 13\,x_3 = 9 \\ \quad\quad x_2 + 5\,x_3 = 3 \end{cases}$$

Solution:

The augmented matrix is $A' = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 6 \\ 2 & 5 & 13 & 9 \\ 0 & 1 & 5 & 3 \end{array}\right]$.

After using Gauss-Jordan reduction we get: $\left[\begin{array}{ccc|c} 1 & 0 & -6 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

i.e.:

$$\begin{cases} x_1 - 6\,x_3 = -3 \\ x_2 + 5\,x_3 = 3 \end{cases}$$

The solution is $x_1 = -3 + 6\,x_3$, $x_2 = 3 - 5\,x_3$, or:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}.$$

5. Find a basis for the null space of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & -1 & 1 \\ 0 & 3 & 2 & 5 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

Solution:

The null space of A is the set of solutions of $A\mathbf{x} = \mathbf{0}$. Gauss-Jordan reduction on A yields: $\begin{bmatrix} 1 & 0 & 5/3 & 2/3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, hence the general solution is

$x_1 = -5x_3/3 - 2x_4/3$, $x_2 = -2x_3/3 - 5x_4/3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2/3 \\ -5/3 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a basis for the null space is:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} -5/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2/3 \\ -5/3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Other solutions are possible.

6. Compute the determinant of the following matrix: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 3 \\ 1 & 1 & 4 & 9 \\ -1 & 1 & 8 & 27 \end{bmatrix}$.

Solution:

$$\det A = 48.$$

7. In $\mathbb{R}^3$ find the change of basis matrix from the standard basis to the basis

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v_3} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

What are the new coordinates of the point $(1, 1, 1)$?

Solution:

The change of basis matrix is $P = [\, \mathbf{v_1}\, \mathbf{v_2}\, \mathbf{v_3} \,] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$

The new coordinates of the point $(1, 1, 1)$ are (x_1, x_2, x_3), where

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

After solving this system we get $x_1 = -2$, $x_2 = 0$, $x_3 = 1$, hence the new coordinates are $(-2, 0, 1)$.

8. Diagonalize the following matrix: $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

Show the matrix P such that $D = P^{-1} A P$ is diagonal.

Solution:

First we find the eigenvalues of A:

$$\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 2)(\lambda - 1)^2$$

hence the eigenvalues are $\lambda = 2$ and $\lambda = 1$ (double).

For $\lambda = 2$ we must solve $(A - 2I)\,\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $x_1 = x_3$, $x_2 = x_3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

so we take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the first eigenvector.

For $\lambda = 1$ we must solve $(A - I)\,\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is $x_1 = -x_2 + x_3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so we take $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the two remaining eigenvectors.

Then[1]

$$P = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned} D = P^{-1}\,A\,P &= \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

[1] Other solutions are possible.

9. Find the principal axes and classify the central conic:

$$x^2 + y^2 - 10\,xy = 24$$

Solution:

The conic can be represented as $\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 24$, where $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$.

We must diagonalize A as $D = P^t A P$ for some *orthogonal* matrix $P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix}$, where $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A.

The eigenvalues of A are the roots of the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\,\lambda - 24 = (\lambda - 6)\,(\lambda + 4)$$

The eigenvalues are $\lambda = -4$ and $\lambda = 6$.

For $\lambda = -4$ we must solve $\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $x_1 = x_2$, or: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so we take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvector.

For $\lambda = 6$ we must solve $\begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $x_1 = -x_2$, or: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so we take $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that $\mathbf{v_1}$ and $\mathbf{v_2}$ are already orthogonal, so all we need is to normalize them: $\mathbf{u_1} = \frac{1}{\sqrt{2}} \mathbf{v_1}$, $\mathbf{u_2} = \frac{1}{\sqrt{2}} \mathbf{v_2}$. The matrix for the change of basis is:

$$P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

In the new basis the conic is $\begin{bmatrix} x' & y' \end{bmatrix} D \begin{bmatrix} x' \\ y' \end{bmatrix} = 24$, where

$$D = P^t\,A\,P = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix},$$

and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^t \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.:

$$\begin{cases} x' = \frac{1}{\sqrt{2}}\,(x+y) \\ y' = \frac{1}{\sqrt{2}}\,(-x+y) \end{cases}$$

Hence the conic is $-4\,x'^2 + 6\,y'^2 = 24$, or equivalently: $-\dfrac{x'^2}{6} + \dfrac{y'^2}{4} = 1$, which is an *hyperbola*. Its principal axes are given by the basic vectors

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Note: An alternative solution is $\dfrac{x'^2}{4} - \dfrac{y'^2}{6} = 1$, and

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

10. Find the maximum and minimum values of the function $f(x, y) = x^2+xy+y^2$ given the constrain $g(x, y) = x^2 + y^2 = 1$.

Solution:

By the method of Lagrange multipliers we must solve $\nabla f(x, y) = \lambda\nabla g(x, y)$, $g(x, y) = 1$, i.e.:

$$\begin{cases} 2x + y = \lambda(2x) \\ x + 2y = \lambda(2y) \\ x^2 + y^2 = 1 \end{cases}$$

which is equivalent to $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} = 1$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. So the problem reduces to finding eigenvectors of length 1 for the matrix A.

First we find the eigenvalues for A:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3),$$

hence the eigenvalues are $\lambda = 1$ and $\lambda = 3$.

For $\lambda = 1$ we solve $(A - I)\,\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$. A solution of length 1 is $\mathbf{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

For $\lambda = 3$ we solve $(A - 3I)\,\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$. A solution of length 1 is $\mathbf{x_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The extreme values of $f(x, y)$ are respectively:

$$f(\mathbf{x_1}) = f(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = 1/2$$

and

$$f(\mathbf{x_2}) = f(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = 3/2\,.$$