Math B17 - Fall 1999 - Midterm Exam No. 2 (solutions)

SOLUTIONS

1. Find the inverse of the following matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{reduction}]{\text{Gauss-Jordan}} \left[\begin{array}{ccc|rrr} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right].$$

Hence: $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

2. Solve the following system of equations:

$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & -3 \\ x_1 & & & + & 2\,x_3 & = & -2 \\ x_1 & + & 2\,x_2 & + & x_3 & = & 0 \end{array}\right.$$

Solution:

The augmented matrix is: $\left[\begin{array}{rrr|r} 1 & -1 & 1 & -3 \\ 1 & 0 & 2 & -2 \\ 1 & 2 & 1 & 0 \end{array}\right]$

After using Gauss-Jordan reduction we get: $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

i.e.:

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

3. Find the rank of the following matrix and a basis for its *column space* (the subspace of $\mathbb{R}^4$ spanned by the columns of A):

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 \\ 2 & 4 & -2 & 1 & 3 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 & 2 \end{bmatrix}$$

Solution:

By using Gauss reduction we get:

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 1 \\ 2 & 4 & -2 & 1 & 3 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 & 2 \end{bmatrix} \overset{\substack{\text{Gauss} \\ \text{reduction}}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We get two pivots, so $\operatorname{rank} A = 2$. The pivots are in the columns 1 and 4, hence the same columns of A form a basis for the column space:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

4. Find a basis and the dimension for the *null space* (set of solutions of $A\,\mathbf{x} = \mathbf{0}$) of the matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 3 & 4 \\ 1 & 0 & -1 & 3 & 5 \end{bmatrix}.$$

Solution:

We have to solve $A\,\mathbf{x} = \mathbf{0}$. Using Gauss-Jordan reduction we get:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 & 4 & 0 \\ 1 & 0 & -1 & 3 & 5 & 0 \end{array}\right] \underset{\text{reduction}}{\overset{\text{Gauss-Jordan}}{\Longrightarrow}} \left[\begin{array}{rrrrr|r} 1 & 0 & -1 & 3 & 5 & 0 \\ 0 & 1 & 0 & -3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

i.e.:

$$\begin{cases} x_1 \qquad - x_3 + 3\,x_4 + 5\,x_5 = 0 \\ \qquad x_2 \qquad - 3\,x_4 - 4\,x_5 = 0 \end{cases}$$

The solution is:

$$\begin{aligned} x_1 &= x_3 - 3\,x_4 - 5\,x_5 \\ x_2 &= 3\,x_4 + 4\,x_5 \end{aligned}$$

i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the following is a basis for the null space of A:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{v_2} = \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v_3} = \begin{bmatrix} -5 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and the dimension is 3.

5. Find the value of p such that the vector $\mathbf{v} = \begin{bmatrix} 3 \\ p \\ 1 \end{bmatrix}$ is in the subspace of $\mathbb{R}^3$ spanned by the vectors:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Find the coordinates of $\mathbf{v}$ respect to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

Solution:

Since $\mathbf{v}$ must be a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, the following system must have solution:

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ p \\ 1 \end{bmatrix}$$

After using Gauss reduction we get:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 2 & -p \\ 1 & 1 & 1 \end{array}\right] \overset{\text{Gauss reduction}}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 4 & p+3 \\ 0 & 0 & -5+p \end{array}\right]$$

The rank of the coefficient matrix is 2. The system will have solution if the augmented matrix also has rank 2. This means that the extra pivot $-5+p$ must be zero, which implies:

$$p = 5\,.$$

The system becomes:

$$\left\{ \begin{array}{rcrcl} x_1 & + & 2\,x_2 & = & 3 \\ & & 4\,x_2 & = & 8 \end{array} \right.$$

Solving by backsubstitution we get the coordinates of $\mathbf{v}$ respect to $\{\mathbf{v_1}, \mathbf{v_2}\}$: $x_1 = -1, x_2 = 2$.

6. Diagonalize the following matrix: $A = \begin{bmatrix} 8 & -5 \\ 10 & -7 \end{bmatrix}$. Find the matrix P such that $D = P^{-1} A P$ is diagonal.

Solution:

The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 8-\lambda & -5 \\ 10 & -7-\lambda \end{bmatrix} = \lambda^2 - \lambda - 6$$
$$= (\lambda - 3)(\lambda + 2)$$

Its roots are $\lambda = 3$ and $\lambda = -2$.

For $\lambda = 3$ we get $A - 3I = \begin{bmatrix} 5 & -5 \\ 10 & -10 \end{bmatrix}$.

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

The solutions of $(A - 3I)\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So we can take the following eigenvector: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, or any non-zero multiple of it.

For $\lambda = -2$ we get $A + 2I = \begin{bmatrix} 10 & -5 \\ 10 & -5 \end{bmatrix}$.

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$.

The solutions of $(A + 2I)\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} x_2/2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So we can take the following eigenvector: $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, or any non-zero multiple of it.

Hence:

$$P = [\mathbf{v_1}, \mathbf{v_2}] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

$$D = P^{-1}\, A\, P = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$